Unit III - Linear Mappings

- linear map(ping)s, transformations, operators
- kernel and image of a linear mapping
  - rank and nullity
- matrix transformations
- non-singular and invertible maps
- matrix representation of a general linear map
- change of basis
- similarity relationship

Several matrix threads converging....

We use matrices in three separate [but related] contexts that must be kept clear....

1. As vectors in the vector space $M_{m,n}$ of matrix addition and scalar multiplication
2. To generate row and column spaces
   - examining linear independence of vectors in R^n
   - in applications related to linear systems [last unit]
3. To define and/or represent linear mappings between vector spaces
   - this application motivates the matrix product definition
   - we study linear mappings in this unit....

The basic definition

- V & U are vector spaces over the same scalars
- a function f: V→U is a *linear mapping* if it preserves the vector space operations
- this means that for v,w∈V and scalar k ...

  f(v+w) = f(v) + f(w)

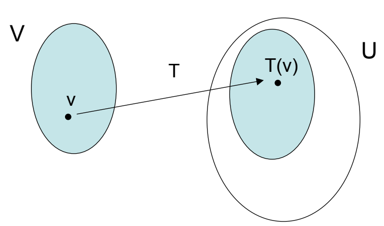
  f(kv) = kf(v)
- alternative terminology
  - linear *transformation* [particularly when f: $R^n \to R^m$]
  - linear *operator* when f: V → V [same v.s.]
  - linear *map*
  - linear *function*

Example: linear maps

[Problem 5.9] Show that T: $R^2 \to R^2$ defined by T(x,y) = (x+y, x) is a linear map.

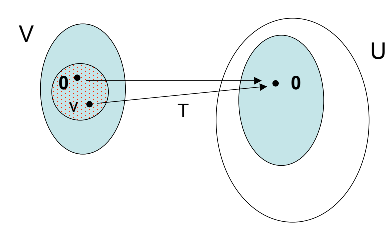
Image of a linear map

- the *image (range)* of T is a subspace of U ....
  T(V) = im(T) = {u∈U | u = T(v) for some v ∈V}



Kernel of a linear map

- for any linear map $T(\mathbf{0}_V) = \mathbf{0}_U$
  - i.e. the zero vector in V maps to the zero vector in U]
- the *kernel* of T is a subspace of V ....
  ker(T) = {v∈V | T(v) = **0**}



Some important linear maps

- the *zero map* T_0:V→U defined by $T_0(v)$ = **0**
- the *identity operator* T_I :V→V defined by $T_I(v)$ = v
- a *dilation* [k>1] and *contraction* [0<k<1] *operator* T :V→V defined by T(v) = kv
  - re-visit this idea when we've studied eigenvalues [unit IV]
- a *projection operator* T: R^n→W = a subspace of R^n
  - R^3 onto the xy-plane defined by T(x,y,z) = (x,y,0)
  - R^3 onto the z-axis defined by T(x,y,z) = (0,0,z)

Some other important linear maps

- a *reflection operator* T: R^n → R^n
  - e.g. T(x,y,z) = (-x,y,z) reflection in the yz-plane
- a *rotation operator* T: R^2 → R^2
  - T(x,y) = (x cosθ - y sinθ, x sinθ + y cosθ) rotates (x,y) by angle θ counter-clockwise
- if B is a basis of V the *coordinate representation* is T: V→ R^n defined by T(v) = $[v]_B$ = the coordinates of v with respect to B

Examples: linear maps of function spaces

- T: P_n→P_{n+1} defined by (T(p))(t) = t p(t) is a linear map
- the *differentiation map* D: C^1[-∞,∞] → *F* [-∞,∞] defined by (D(f))(x) = f'(x) the derivative function
- the *integration map* J: C^0[-∞,∞] → C^1 [-∞,∞] defined

  by (J(f))(x) = $\int_0^x f(t)dt$

Non-examples: map that are NOT linear

- T: M_{nn}→R defined by T(A) = det(A)
  - not linear because det(A+B) ≠ det A + det B
- T: R^3 → R defined by T(x,y,z) = $x^2+y^2+z^2$
  - not linear because T(kv) = k^2T(v)
- a translation operator T: R^2 → R^2 defined by T(x,y) = (x+2,y-5)
  - not linear because T(0) = (2,-5) ≠ (0,0)
- the operator T: R^2 → R^2 defined by T(x,y) = (xy,x)
  - not linear because T(v+w) ≠ T(v) + T(w)

Composition of mappings

- let T_1: V → U and T_2: U → W be linear maps
- the *composition* of the two maps is the map $T_2 \circ T_1 : V \to W$ defined by $(T_2 \circ T_1)(v) = T_2(T_1(v))$
- the composition of two linear maps is a linear map

Matrix transformations

- for any m×n matrix A we can define a *matrix transformation* T_A: R^n → R^m by the matrix product $T_A(v)$ = Av
  - the identity map corresponds to the identity matrix
  - the zero map corresponds to the zero matrix
- let A be m×p and B be p×n matrices
- the composition of the two linear transformations T_A: R^p → R^m and T_B: R^n → R^p is T_{AB}: R^n → R^m
- so composition of linear transformations corresponds to the product of their matrices

$$T_A \circ T_B : R^n \to R^m$$

Example: matrix transformations

[Problem 5.10] Use matrices to show that T: $R^3 \to R^2$ defined by T(x,y,z) = (x+y+z, 2x-3y+4z) is a linear map.

Using bases to define transformations

- let $\{v_1, \ldots v_n\}$ be a <u>basis</u> of V and $\{u_1, \ldots u_n\}$ <u>any</u> vectors in U
- define a mapping T: V → U by

  $$T(v) = a_1u_1 + \ldots + a_nu_n$$

  where $v = a_1v_1 + \ldots + a_nv_n$ is expressed in terms of the given basis
- this completely characterizes a **<u>unique linear map</u>** T: V → U in terms of its action on a basis:

  $$\mathbf{T(v_1) = u_1, T(v_2) = u_2, \ldots, T(v_n) = u_n}$$
- to show this take $v = a_1v_1 + \ldots + a_nv_n$ and $w = b_1v_1 + \ldots + b_nv_n$
- we have....

....Using bases to define transformations

- $T(v+w) = T[(a_1+ b_1)v_1 + \ldots + (a_n+ b_n)v_n]$
  $= (a_1+ b_1)u_1 + \ldots + (a_n+ b_n)u_n$
  $= (a_1u_1 + \ldots + a_nu_n) + (b_1u_1 + \ldots + b_nu_n)$
  $= T(v) + T(w)$
- $T(kv) = T(ka_1v_1 + \ldots + ka_nv_n)$
  $= ka_1u_1 + \ldots + ka_nu_n$
  $= k(a_1u_1 + \ldots + a_nu_n)$
  $= kT(v)$
- so the mapping is <u>linear</u>
- it is also easy to show that putting $T(v_i) = u_i$ defines a unique map T [see problem 5.13]

Example: Using bases to define a linear map

[Problem 5.14] Find the unique linear map T: $R^2 \to R^2$ so that T(1,2) = (2,3) and T(0,1) = (1,4).

Example: Using bases

(a) Find the unique linear map T: $R^3 \to R^2$ so that T(1,1,1) = (1,0), T(1,1,0) = (2,-1), T(1,0,0) = (4,3) [Translation: Find a formula for T(x,y,z)] (b) Evaluate T(2,-3,5).

.....Example: Using bases

The kernel and image of a linear map

- T: V → U is a linear map ....
- T(V) is a subspace of U .... [the image of T]
- ker(T) is a subspace of V .... [the kernel of T]

Don't get confused about this point!

Finding the image of a linear map

- if $\{v_1, ..., v_n\}$ span V and T: V → U is a linear map then $\{T(v_1), ..., T(v_n)\}$ span the image T(V)
  - let $u \in T(V)$
  - then $u = T(v)$ for some $v \in V$
  - but $v = a_1v_1 + ... + a_nv_n$
  - so $u = T(a_1v_1 + ... + a_nv_n) = a_1T(v_1) + ... + a_nT(v_n)$
- so a **linear** map preserves **linear** combinations of vectors

Finding the image of a matrix transformation

- **for a matrix transformation $T_A : R^n \to R^m$ the image $T_A(R^n)$ is precisely the column space of A**
  - consider R^3 and take the standard basis vectors e_1, e_2, e_3

$$T_A(e_2) = T_A(0,1,0) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$$

  - so $T_A(e_2) = C_2$ the second column of A
  - in general $T_A(e_i) = C_i$ the ith column of A
  - since the standard basis vectors span R^n the columns of A must span the image $T_A(R^n)$ as per slide 20 result

The rank of a linear map

- the *rank* of a linear map T: V → U is the dimension of its image T(V)
- if $T_A: R^n \to R^m$ is a matrix transformation the rank of T_A is the rank of the matrix A
  - $T_A(R^n)$ is just the column space of A [slide 21] so the two concepts coincide

Example: image of a linear map

[Text example 5.9] find a basis for the image of T: $R^4 \to R^3$ defined by T(x,y,z,t) = (x-y+z+t, 2x-2y+3z+4t, 3x-3y+4z+5t)

Finding the image of a linear map

- this example illustrates something important
- even though a linear map preserves linear combinations of vectors ....
- .... a linear map does NOT preserve linear independence in general
- so a basis of V does NOT necessarily map to a basis of the image T(V)

Example: image of a linear map

[Problem 5.16] Find a basis for the image of T: $R^4 \rightarrow R^3$ defined by T(x,y,z,t) = (x-y+z+t, x+2z-t, x+y+3z-3t)

Example: image of a linear map

[Problem 5.17] find a basis for the image of T: $R^3 \rightarrow R^3$ defined by T(x,y,z) = (x+2y-z, y+z, x+y-2z)

The nullity of a linear map

- the *nullity* of a linear map T: V → U is the dimension of its kernel ker(T)
- if T_A: $R^n \rightarrow R^m$ is a matrix transformation the kernel ker(T_A) is called the *nullspace* of A
- ker(T_A) consists of all vectors v for which Av = 0.... so to find a basis for ker(T_A) you just solve this system of linear equations

A **VERY** important result

- for a linear map T: V→U we have

  rank T + nullity T = dim V
- for a matrix transformation T_A : $R^n \rightarrow R^m$ this translates to a matrix result about A

  rank A + nullity A = n

These relationships are central to all of linear algebra

Examples: kernel of a linear map

[Text example 5.9] find a basis for the kernel of T: $R^4 \rightarrow R^3$ defined by T(x,y,z,t) = (x-y+z+t, 2x-2y+3z+4t, 3x-3y+4z+5t)

Examples: kernel of a linear map

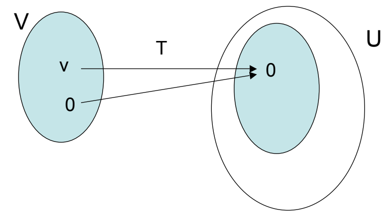
[Problem 5.16] Find a basis for the kernel of T: $R^4 \rightarrow R^3$ defined by T(x,y,z,t) = (x-y+z+t, x+2z-t, x+y+3z-3t)

Examples: kernel of a linear map

[Problem 5.17] Find a basis for the kernel of T: $R^3 \rightarrow R^3$ defined by $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$

Non-singular maps

- a map T: V → U is *non-singular* if ker T = {0}
- equivalently a map is singular if the image of some non-zero vector is zero as shown below....

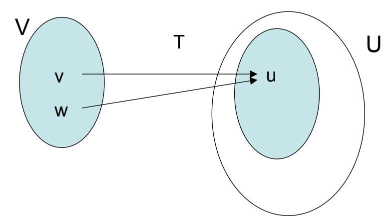


Illustrative examples

- examples of singular maps:
  - the zero map is singular
  - a projection map is singular, e.g. $T(x,y,z) = (x,y,0)$ because $T(0,0,z) = (0,0,0)$ for any z
  - the differentiation operator D on P_n is singular because all constant polynomials get mapped to the zero polynomial
  - the maps in example 5.9 and problems 5.16&5.17 [slides 30-31] are all singular because they have non-zero kernels
- examples of non-singular maps:
  - the identity, dilation, contraction, and rotation maps are all non-singular
  - the map T: $R^2 \rightarrow R^3$ defined by $T(x,y) = (x+y, x-y, x+y)$ is non-singular [$T(x,y) = (0,0,0)$ implies $(x,y) = (0,0)$]
  - **any matrix transformation T_A : $R^n \rightarrow R^m$ for which rank A = n is non-singular [why?]**

One-to-one maps

- a map T: V → U is *one-to-one* if T(v) = T(w) implies v = w
- obviously a singular map cannot be one-to-one but...not all non-singular maps are one-to-one
- maps as shown below are <u>not</u> one-to-one

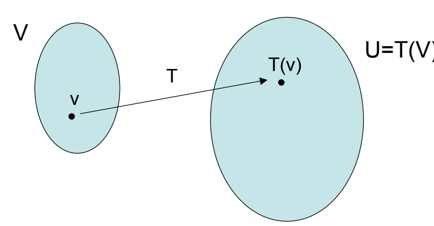


Non-singular linear maps

- a <u>linear</u> map T: V → U is one-to-one if and only if it is non-singular
  - T one-to-one implies only the zero vector maps to the zero vector of U
  - T non-singular and T(v) = T(w) implies T(v - w) = T(v) - T(w) = 0 so v - w = 0 or v = w
- a non-singular <u>linear</u> map T: V → U preserves the linear independence of vectors in the images
  - suppose $v_1, v_2, \ldots, v_n$ are linearly independent
  - suppose $a_1T(v_1) + \ldots + a_nT(v_n) = 0$
  - T is linear so $T(a_1v_1 + \ldots + a_nv_n) = 0$
  - T is non-singular so $a_1v_1 + \ldots + a_nv_n = 0$
  - but $v_1, v_2, \ldots, v_n$ are linearly independent so all $a_i = 0$

Onto maps

- a map T: V → U is *onto* if the range T(V) = U
- in other words 'the images of vectors in V cover the entire space U'



Onto linear maps

- a non-singular linear map T: V → U is automatically one-to-one but it may not be onto....
- example
  - define the map $T: P_n \to P_{n+1}$ as 'multiplication by t' [slide 9]
  - $T(a_0 + a_1t + a_2t^2 + ... + a_nt^n) = a_0t + a_1t^2 + a_2t^3 + ... + a_nt^{n+1}$
  - T is a non-singular linear map but.....
  - T is not onto because the constant polynomials in P_{n+1} are not in the image of T
- in general for T: V → U if dim U > dim V it is possible to define one-to-one (non-singular) maps that aren't onto
- the problem here is that there is 'more room in U than V' so there can be vectors in U not in T(V)

Non-singular linear maps again

- the situation is fixed when U and V have the same finite dimension.....
- provided dim V = dim U a linear map T: V → U is onto if and only if it is non-singular
  - if T is onto then T(V) = U ⇒ rank T = dim U = n say, so nullity T = dim V - rank T = n - n = 0 so ker T = {0}, i.e. T is non-singular
  - if T is non-singular then nullity T = 0 ⇒ rank T = dim V - nullity T = n - 0 = n, i.e. dim T(V) = n = dim U, so we must have T(V) = U [why?], i.e. T is onto

Non-singular linear operators

- in particular, for linear operators T: V → V all of the following are equivalent
  - T is non-singular
  - T is one-to-one
  - T is onto
- further in particular, these results apply in the case of matrix transformations $T_A: R^n \to R^n$, i.e. for which A is square...all of the following are equivalent
  - Av = 0 has only the zero solution v = 0 [non-singular]
  - any vector $b \in R^n$ can be written b = Ax for some $x \in R^n$ [onto]
  - rank A = n [onto]

Invertible maps

- in order to invert a map T: V → U we need to define a map T^{-1}: U → V which reverses T
- defining T^{-1} in this way by un-doing the action of T requires that all vectors in U must be images of
  - some vector in V [T is onto]
  - only one vector in V [T is one-to-one]
- i.e. to be invertible the map has to be one-to-one and onto U [called an *isomorphism*]
- formal definition: a map T: V → U is *invertible* if there is another map T^{-1}: U → V so that $T^{-1} \circ T$ is the identity operator on V and $T \circ T^{-1}$ is the identity operator on U

Invertible matrix transformations

- if the linear map is a matrix transformation $T_A: R^n \to R^m$ it is invertible precisely when the matrix A is invertible
- the inverse map is defined by $T_A^{-1} = T_{A^{-1}}$
- A must be a square matrix [slides 38-39]
- so a square matrix is invertible if and only if it is non-singular...the two concepts coincide

Example: non-singular non-invertible linear map

[Problem 5.26] T: $R^2 \to R^3$ defined by T(x,y) = (x+y, x-2y, 3x+y).

Example: invertible matrix transformation

[Problem 5.40] Is the operator T: $R^3 \to R^3$ defined by
T(x,y,z) = (2x, 4x-y, 2x+3y-z) invertible? Find the inverse.

Plaine English summary 1

- T: V → U any old map
- a non-singular map
  - no non-zero vector maps to the zero vector
- a one-to-one map
  - two different vectors don't map to the same image vector
- an onto map
  - there are no vectors in U that aren't the image of some vector in V
- an isomorphism
  - both one-to-one and onto
- an invertible map
  - must be an isomorphism to define an inverse map

Plaine English summary 2

- a linear map T: V → U
  - preserves linear combinations
  - a spanning set for V maps to a spanning set for the image
  - is completely characterized by its action on a basis of V
- a non-singular linear map
  - must also automatically be one-to-one
  - preserves the linear independence of vectors in the image vectors
- a non-singular linear map with dim U = dim V
  - must also automatically be onto
  - is an isomorphism
  - is invertible
- for linear operators all of the terms are equivalent

Matrix representations: to do list

1. How do we construct a matrix representation A for a general linear transformation T:V→W?
   - we'll restrict to the case of a linear operator where T:V→V, but only to keep things a little simple
   - in this case the matrix A will be square and....
   - T(v) = A[v] where $[v] \in R^n$ are coordinates of $v \in V$
2. How do the coordinates of a vector change when we use a different basis?
3. How does the matrix representation of a linear operator change if we change the basis?

1. Matrix representation of a linear operator

- let V be a finite-dimensional v.s.
- choose a basis $\beta = \{u_1, ..., u_n\}$ of V..notation alert :-(
- a linear operator T:V→V is completely determined by its action on a basis of V: $T(u_1), ..., T(u_n)$
- these are vectors in V, so they can be expressed in terms of the basis β:

  $T(u_1) = a_{11}u_1 + a_{12}u_2 + ... + a_{1n}u_n$
  $T(u_2) = a_{21}u_1 + a_{22}u_2 + ... + a_{2n}u_n$
  . . . .
  $T(u_n) = a_{n1}u_1 + a_{n2}u_2 + ... + a_{nn}u_n$
- recall the coordinates of $T(u_i)$ with respect to β are just $[T(u_i)]_\beta = [a_{i1}, a_{i2}, ..., a_{in}]_\beta$

Matrix representation of a linear operator

- arrange these coordinate vectors as the columns of a matrix: $[T]_\beta = \left[\ [T(u_1)]_\beta \mid [T(u_2)]_\beta \mid \cdots \mid [T(u_n)]_\beta\ \right]$

$$= \left[\begin{array}{c|c|c|c} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{array}\right]$$

- this is called the *matrix representation* of T with respect to the basis β
- we can use this matrix $[T]_\beta$ and coordinate vectors in R^n instead of T and vectors in V because:

  $$\mathbf{[T(v)]_\beta = [T]_\beta [v]_\beta}$$

Example: Matrix representation in R^2

[Problem 6.2] $T(x,y) = (2x-7y, 4x+3y)$, $\beta = \{u_1,u_2\} = \{(1,3), (2,5)\}$. Find $[T]_\beta$ and verify that $[T]_\beta [v]_\beta = [T(v)]_\beta$ for $v = (4,-3)$.

Finding matrix representations - method

- we restrict to a linear <u>operator</u> $T:V \to V$
  - the matrix respresentation has to be square [why?]
- to find a matrix representation of T with respect to a basis $\beta = \{u_1,u_2, \ldots , u_n\}$ of V
  - find a formula to express any vector $v \in V$ as a linear combination of basis vectors $u_1,u_2, \ldots , u_n$ [convenient]
  - for each basis vector u_k find $T(u_k)$ and express it as a linear combination of basis vectors $u_1,u_2, \ldots , u_n$
  - arrange these coordinates for $T(u_k)$ as the columns of a matrix $[T]_\beta$
  - this $[T]_\beta$ is the required matrix representation of T with respect to β

Example: Matrix representation in R^3

[Problem 6.5] $T(x,y,z) = (2y+z, x-4y, 3x)$, $\beta = \{u_1, u_2, u_3\} = \{(1,1,1), (1,1,0), (1,0,0)\}$. Find $[T]_\beta$ and verify that $[T]_\beta [v]_\beta = [T(v)]_\beta$ for any v.

.....Example: problem 6.5 (cont'd)

Connection with matrix operators

- if T_A is a matrix transformation on R^n the matrix representation of T_A with respect to the <u>standard</u> basis $\varepsilon = \{e_1, e_2, \ldots, e_n\}$ is simply the matrix A
- in fancy notation we have $[T_A]_\varepsilon [v]_\varepsilon = [Av]_\varepsilon$ for $v \in R^n$.... or just $[T_A] v = Av$ where the standard basis is understood [assumed]
- in general you can drop the subscript that tells you the basis if it is obvious....but if any doubt it's best for clarity to...
  - leave the subscript in, or
  - write out the linear combinations of basis vectors explicitly as in the text solutions [e.g. problems 6.5,6.6 etc]
- whenever we write "T_A" and give the matrix A we are <u>assuming the standard basis</u> is being used

Example: Matrix operator in R^3

[Problem 6.6] T_A is a linear operator on R^3 defined by the matrix:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 0 \\ 1 & 4 & -2 \end{bmatrix}$$

$\beta = \{u_1, u_2, u_3\} = \{(1,1,1), (0,1,1), (1,2,3)\}$ is a basis of R^3. Find $[T_A]_\beta$ and verify that $[T_A]_\beta [v]_\beta = [T_A(v)]_\beta = [Av]_\beta$ for any v.

....Example: problem 6.6 cont'd

2. Changing basis

- suppose we have two different bases for V:
  - the old basis $\beta = \{u_1, u_2, \ldots, u_n\}$ and...
  - the new basis $\beta' = \{v_1, v_2, \ldots, v_n\}$
- since β is a basis each new basis vector v_i can be expressed as a unique l.c. of old basis vectors in β:

$$v_1 = a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n$$
$$v_2 = a_{21}u_1 + a_{22}u_2 + \ldots + a_{2n}u_n$$
$$\ldots\ldots\ldots$$
$$v_n = a_{n1}u_1 + a_{n2}u_2 + \ldots + a_{nn}u_n$$

- recall the coordinates of each new v_i with respect to the old basis β are just $[v_i]_\beta = [a_{i1}, a_{i2}, \ldots, a_{in}]_\beta$

Changing basis

- arrange these coordinate vectors as the columns of a matrix: $P = \left[\ [v_1]_\beta \mid [v_2]_\beta \mid \cdots \mid [v_n]_\beta\ \right]$

$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

- P must be invertible [why?]
- P gives a formula to change the coordinates of a vector from the new basis back to the old basis:

$$[w]_\beta = P[w]_{\beta'}$$

- to go the other way [new from old] we use P^{-1}:

$$[w]_{\beta'} = P^{-1}[w]_\beta$$

Terminology alert

- P expresses the new basis vectors in terms of the old ... hence its name "change of basis matrix from old to new" but....
- ... it's P^{-1} that converts coordinates in the old basis into coordinates expressed in the new basis
- unfortunately this terminology is not standard and can be confusing
- so be careful!
- the change of basis matrix P from the standard basis to a new basis β consists of the vectors of β arranged as columns
  - this works **only** when the old basis is the standard basis

Example: changing basis from standard

[Example 6.6-6.7] Consider the standard basis $\varepsilon = (e_1, e_2, e_3)$ and a new basis $\beta = \{u_1, u_2, u_3\} = \{(1,0,1), (2,1,2), (1,2,2)\}$. Find (a) the change of basis matrix P from ε to β and vice versa (b) the coordinates of the vector (1,3,5) with respect to the new basis.

..... Example: example 6.6-6.7 (cont'd)

Changing basis - summary of method

- to change basis between ...
  old basis $\beta = \{u_1, u_2, \dots, u_n\}$ and
  new basis $\beta' = \{v_1, v_2, \dots, v_n\}$
  - find a formula to express any vector $v \in V$ as a linear combination of basis vectors $u_1, u_2, \dots, u_n$ [convenient]
  - for each basis vector v_k find the β coordinates $[v_k]_\beta$
  - arrange these β coordinates for the v_k's as the columns of the matrix P
  - the inverse P^{-1} is the matrix that converts old β coordinates into new β' coordinates:

$$[w]_{\beta'} = P^{-1}[w]_\beta$$

Example: changing basis in R^2

[problem 6.17] Old basis $\beta = \{u_1, u_2\} = \{(1,-2), (3,-4)\}$ and new basis $\beta' = \{v_1, v_2\} = \{(1,3), (3,8)\}$. Find the change of basis matrix P and verify that $P[w]_{\beta'} = [w]_\beta$ for any vector w.

.....Example: problem 6.17 cont'd

3. Changing basis in a matrix representation

- bases old β and new β' & change of basis matrix P
- for a linear operator $T: V \to V$ the matrix representation with respect to the new basis β' is

$$[\,T\,]_{\beta'} = P^{-1}[\,T\,]_\beta P$$

- if A and B are matrix representations of T with respect to different bases then there is an invertible matrix P so that

$$B = P^{-1}AP$$

- matrices related in this way are called *similar*
- this procedure is often very useful in practice to find a particularly simple form of the matrix representation, e.g. a diagonal matrix [Unit IV]

Why does the change of basis formula work?

- let $v \in V$ be any vector. Then we have:

$$\begin{aligned} P^{-1}[T]_\beta P \bullet [v]_{\beta'} &= P^{-1}[T]_\beta \bullet P[v]_{\beta'} \\ &= P^{-1}[T]_\beta \bullet [v]_\beta \\ &= P^{-1} \bullet [T]_\beta [v]_\beta \\ &= P^{-1} \bullet [T(v)]_\beta \\ &= [T(v)]_{\beta'} \\ &= [T]_{\beta'} \bullet [v]_{\beta'} \end{aligned}$$

- so $P^{-1}[T]_\beta P = [T]_{\beta'}$ as required

Example: linear operators and change of basis

[problem 6.23] The linear operator T is defined on R^2 by the formula $T(x,y) = (5x-y, 2x+y)$. The old basis is the standard basis ε and the new basis is $\beta = \{u_1, u_2\} = \{(1,4), (2,7)\}$. Find (a) the two change of basis matrices P and P^{-1}, (b) $[\,T\,]_\varepsilon$ and (c) $[\,T\,]_\beta$

.... Example: problem 6.23 cont'd

Changing basis in a matrix representation-method

- old basis $\beta = \{u_1, u_2, \ldots, u_n\}$ and new basis $\beta' = \{v_1, v_2, \ldots, v_n\}$
- T: V→V is a linear operator with matrix representation $[T]_\beta$
- to find the new matrix representation $[T]_{\beta'}$ of T with respect to β' you can
  - find the β coordinates $[T(v_k)]_\beta$ for each new basis vector and follow the procedure on slide 50

    OR [easier].....
  - find the change of basis matrix P and use the formula:

    $$[\,T\,]_{\beta'} = P^{-1}[\,T\,]_\beta P$$

Example: matrix representations change of basis

[problem 6.25] The linear operator T is defined on R^3 by the formula T(x,y,z) = (x+3y+z, 2x+5y-4z, x-2y+2z). Find the matrix B which represents T with respect to the basis $\beta = \{u_1, u_2, u_3\}$ = {(1,1,0), (0,1,1), (1,2,2)}.

.... Example:problem 6.25 cont'd

Matrix representations: wrap-up

T: V→V a linear operator. $\beta = \{u_1, \ldots u_n\}$ a basis of V.
$\beta' = \{v_1, \ldots v_n\}$ another basis of V.

1. The <u>matrix representation</u> of T with respect to β is the matrix $[T]_\beta$
   - arrange the coordinate vectors $[T(u_i)]_\beta$ as columns for $[T]_\beta$
   - for any vector w we have $[T(w)]_\beta = [T]_\beta [w]_\beta$
2. The <u>change of basis matrix</u> from β to β' coordinates is the matrix P
   - arrange the coordinates of $[v_i]_\beta$ as columns for P
   - the coordinates of a vector w with respect to the new basis β' are given by $[w]_{\beta'} = P^{-1}[w]_\beta$
3. The matrix representation of T with respect to the new basis β' is given by $[T]_{\beta'} = P^{-1}[T]_\beta P$