

SyDe114 Linear Algebra: Mini-Test 3

DUE: 2 June 2005

Define subspaces of R^5 by

$$\begin{aligned}U &= \{(a, b, c, d, e) : a + 2b + c = 0, 3b - d + e = 0\} \\W &= \{(a, b, c, d, e) : a + b + 2c - d = e\}\end{aligned}$$

(a) Determine the dimension of $U \cap W$ WITHOUT¹ finding a basis for it. (b) Find a basis for $U \cap W$ and confirm your answer in (a).

SOLUTION. (a) U is defined by 2 constraints, which are clearly independent since they are expressed in echelon form. Thus $\dim(U) = 5 - 2 = 3$. W is defined by one constraint, which is trivially independent, and so, $\dim(W) = 5 - 1 = 4$.

Now we need a basis for $U + W$ in order to determine its dimension. We find spanning sets for each of U and W . First for U , the constraints are already in echelon form:

$$\begin{aligned}a + 2b + c &= 0 \\3b - d + e &= 0\end{aligned}$$

so the pivot variables a and b can be expressed in terms of the other free variables. The general vector in U is of the form:

$$\begin{aligned}(-3c - 2d + 2e, d - e, 3c, 3d, 3e) &= c(-3, 0, 3, 0, 0) + d(-2, 1, 0, 3, 0) + e(2, -1, 0, 0, 3) \\&= cu_1 + du_2 + eu_3\end{aligned}$$

giving a basis $\{u_1, u_2, u_3\}$ for U . Second for W , the defining constraint gives the general form of a vector:

$$\begin{aligned}(-b - 2c + d + e, b, c, d, e) &= b(-1, 1, 0, 0, 0) + c(-2, 0, 1, 0, 0) + d(1, 0, 0, 1, 0) + e(1, 0, 0, 0, 1) \\&= bw_1 + cw_2 + dw_3 + ew_4\end{aligned}$$

¹The question wording was actually faulty. Here's an explanation. The intent was that you should produce something like (a) and (b) here and use the dimension formula. However, a short cut is to combine the three constraints, reduce them to echelon form and demonstrate that they are independent, then conclude that $\dim(U \cap W) = 5 - 3 = 2$. In other words, you could have got away without doing everything in (a), using the first part of (b) for the $U \cap W$ answer and the remaining part for the basis finding and confirmation. It could be argued (not by me) that reducing the constraints to echelon form is half way toward finding a basis for $U \cap W$, but you will be given full marks for a proper short cut solution. It's very difficult to word problems unambiguously!

giving a basis $\{w_1, w_2, w_3, w_4\}$ for W . Combining the bases for U and W provides a spanning set for $U + W$. Check linear independence of $\{w_1, w_2, w_3, w_4, u_3\}$ ²:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The row-reduced echelon matrix shows that the five vectors are linearly independent. Therefore $\dim(U + W) = 5$.

Now, to conclude we find:

$$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W) = 3 + 4 - 5 = 2$$

(b) To find a basis for $U \cap W$, we combine the three constraints, express as row vectors (a, b, c, d, e) in a matrix, and reduce to echelon form:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & -1 & 1 \\ 1 & 1 & 2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & -1 & 1 \\ 0 & 0 & 3 & -4 & -2 \end{bmatrix}$$

Thus the first three coordinates can be expressed in terms of the other two (free) coordinates:

$$\begin{aligned} 3c &= 4d + 2e \\ 3b &= d - e \\ 3a &= -6b - 3c = -2d + 2e - 4d - 2e = -6d \end{aligned}$$

where the last equation is written as $3a$ to avoid fractions in the next step. The general vector, that satisfies the three constraints, can be written as:

$$\begin{aligned} (a, b, c, d, e) &= (-6d, d - e, 4d + 2e, 3d, 3e) \\ &= d(-6, 1, 4, 3, 0) + e(0, -1, 2, 0, 3) \\ &= dv_1 + ev_2 \end{aligned}$$

giving a basis $\{v_1, v_2\}$ for $U \cap W$. This shows that $\dim(U \cap W) = 2$, which is consistent with the conclusion in part (a).

²Note the choice of these vectors. You know that the first four vectors are linearly independent, because they are generated from a single constraint equation, so include all of those. There's no need to include all three of the u vectors. Just pick one of them that demonstrates five independent vectors all together