STEPHEN BIRKETT AND WILLIAM JURGENSON

Why Didn't Historical Makers Need Drawings?

Part I – Practical Geometry and Proportion

HISTORICAL DESIGN PRACTICE

In his pioneering work on harpsichord building, Frank Hubbard characterized the historical approach as 'like a beaver building his dam ... the maker constructed his case, guided by experience for the length and by the known size of a keyboard of the projected range for the width.' [1] Hubbard has been criticized for this apparently enigmatic description and for being content to ignore the important details: 'Today we want to know how dimensions were chosen and how these relate to each other.' [2] The implication is that modern organology should take a more sophisticated approach to the reconstruction of historical design practice. The existence of those 'important details' is taken for granted, with perhaps even a secret and complicated lost practice waiting to be rediscovered by a modern analyst. Historical reality suggests, however, that Hubbard's assessment was actually remarkably complete as a description of the fundamental approach, and that the search for filling in the details must relate primarily to the specifics of particular instruments.

In 1624, the architect Henry Wotton succinctly described the three key design elements which any artificer must consider: 'In Architecture as in all other Operative Arts, the end must direct the Operation. The end is to build well. Well building has three conditions. Commoditie, Firmenes, and Delight.' [3] In other words, pragmatism and simplicity were the historical builder's guiding principles as he approached the problem of designing and constructing an artifact. Obscure, complicated procedures, including the incorporation of various mystical proportions or mathematical relationships for no practical reason, as some modern writers have suggested, are inconsistent with this approach. To fill in the missing pieces of Hubbard's 'beaver dam' requires us to look in the right place for the right sort of pieces, rather than following the temptation to hypothesize complicated theories and force these to fit the data.

Even while a craft was still a living tradition it was a difficult problem to acquire knowledge of the working methods. The eighteenth-century encyclopedist Diderot complained that 'craftsmen ... live isolated, obscure, unknown lives; everything they do is done to serve their own interest; they almost never do anything just for the sake of the glory.' [4] He goes on to suggest that, for the most secretive trades, 'the shortest way of gaining the necessary information would be to bind oneself out to some master as an apprentice.' In one respect, our position now is more difficult than Diderot's, because the working methods and design principles and practice are lost

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[1] F. Hubbard. *Three Centuries of Harpsichord Making*. Harvard Univ. Press, Cambridge, Mass., 1965, pp.210-11.

[2] J. Koster. Toward the reconstruction of the Ruckers' geometrical methods. In: C. Rieche, Ed. *Kielinstrumente aus der Werkstatt Ruckers – zu Konzeption, Bauweise und Ravalement sowie Restaurierung und Konservierung: Bericht über die internationale Konferenz vom 13-15 September, 1996 im Händel-Haus, Halle*. Halle, 1998.

[3] Henry Wotton. *The Elements of Architecture collected by Henry Wotton Kt from the best Authors and Examples.* London, 1624. Reprinted Longmans and Green, London, 1903.

[4] D. Diderot and Jean d'Alembert, Eds. *Encyclopédie ou Dictionnaire raisonné des sciences, des arts, et des métiers*. Paris, 1751-1758. Excerpts transl. in: Jacques Barzun, Ralph Bowen. *Rameau's Nephew and Other Works.* Doubleday, NY, 1956.

from a living tradition. Nevertheless, our modern perspective can also give the advantage of a better viewpoint from which to sort things out in an overall historical context.

In the crafts tradition, skills were passed on orally from master to apprentice, therefore one should not expect to find more than a scant body of published source material of any specificity. Furthermore – and on this point many modern analysts go astray – there is likely to have been little direct connection between the pragmatic methods employed by craftsmen, and the contemporary theoretical publications, so, for the most part, these have to be rejected as useful sources of information. Complex algebraic methods based on numerical calculation have no place in the historical context of the craftsman. Despite the lack of appropriate source material for individual crafts, the very strong inter-relationships which existed between them imply that it is reasonable to extrapolate from one to another.[5] Therefore, well-supported general principles can be given considerable weight. In particular, historical source material related to the building crafts can be useful, including late-medieval works which present specific design methods used by master masons, architectural treatises of various Renaissance writers, works on furniture design, and archival material on building projects.

It has been suggested that our modern intellectual apparatus makes it difficult to think about design in the same way that the early builder did,[6] and that our understanding of their methods is still very limited.[7] This is not an accurate assessment. The general principles and practice of historical design are really quite well understood: proportional relationships were emphasized through the exclusive use of simple, pragmatic, yet highly effective geometrical methods, with absolute dimensions fixed by the choice of a single modular dimension. Using evidence from appropriate sources, we establish in this article that these design principles were the basis for the practice of all historical craftsmen, and examine how they were applied, in particular, by makers of stringed keyboard instruments. To do this unequivocally, a lengthy and comprehensive examination of historical design practice in the most general context is required to establish the working framework, before the specifics that pertain to instrument makers can be inferred. To avoid false conclusions, it is very important to eschew a modern bias in this process, by not proposing methods simply because they may seem reasonable.

In this first part of the article we describe the fundamental role played by proportional relationships in historical design, and how these were established by the craftsman's practical geometry using the concept of the module. For a meaningful analysis, for instance of an extant musical instrument, it is critical to work with the correct causality by re-constructing the geometry, rather than merely reporting observed proportions which have no practical context. Almost all published analyses of musical instruments fail in this respect. Geometric design techniques are examined for two distinct types of stringed keyboard instruments, parallel, in which the strings are in the line with the keys, and perpendicular, in which strings and keys are at, or close to, a right angle. Historical techniques were based either on constructive geometry using layout tools – the square, straightedge, and compasses – or on modular measurement, i.e. the transfer of multiples of an elementary modular unit to the artifact. The first of these methods is illustrated by a reconstruction of geometric designs inferred from an analysis of extant five-octave pianos by J.A. Stein and his followers.

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[5] This is well-documented. For instance, Hanns Schmuttermayer, a late fifteenth-century Nürnberg goldsmith, wrote a booklet for masons, explaining in the preface that he had composed it "for the instruction of our fellowmen and all masters and journeymen who use the high and liberal art of geometry." Shelby, Ed. *Gothic Design Techniques. The Fifteenth-Century Design Booklets of Mathes Roriczer and Hanns Schmuttermayer.* Southern Illinois Univ. Press. Feffer and Simons, London and Amsterdam, 1977, p.55-58.

[6] R. Gug. Geometry, lutherie and the art of historiography. Fellowship of the Makers and Restorers of Historical Instruments (FoMRHI) No. 59 (April 1990) 40-72.

[7] D. Wraight. The identification and authentication of Italian string keyboard instruments. In: H. Schott, Ed., *The Historical Harpsichord*, Vol. 3. Pendragon Press, Stuyvesant, NY, 1992, pp.59-161.

MODERN AND HISTORICAL DESIGN PRINCIPLES CONTRASTED

It is worthwhile to consider the problem at an abstract level initially. The fabrication of any artifact requires three conceptually distinct processes: establishing the design by specifying the general form and the relationships between its parts; storage of design parameters for future reference; and transfer of the design to the actual objects being made. A proper reconstruction of the working practices of an historical builder must accommodate and explain how each one of these three processes was accomplished within the context of the period.

The independence of these three fabrication processes is a cornerstone of modern manufacturing practice, in which designs are stored for reference in the form of technical drawings (physical or computer-based blueprints), and these are used repeatedly to establish the desired configuration of coordinates with respect to a 'reference grid' on each object being constructed. Whatever method is used to accomplish this transfer of dimensions – whether direct measurement, templates, dividers, optical devices, computer assistance, and so on – the essence of the modern approach is a one-to-one comparison between dimensions on object and drawing.

Modern engineering methodology was certainly used by stringed keyboard instrument manufacturers toward the latter part of the Nineteenth Century, as documented and described by Julius Blüthner in the *Lehrbuch*[8] for pianobuilders. There seems to be no evidence, however, to support the use of drawings by historical builders prior to the late Nineteenth Century. Based on his extensive observations of instruments and an exhaustive compilation of source material on harpsichord building, Hubbard concludes 'it is remarkable that old makers do not seem to have worked very much from drawings ... In all our inventories there are only three hints of drawings or templates, and none is very specific.' [9] One might think that the difficulty of finding a convenient stable medium for recording a full-scale drawing precluded their general use, but this implies a revisionist way of thinking, i.e. that a builder would have used a seemingly 'better' modern method had it been possible for them to do so. As will be demonstrated, the use of a technical drawing is actually incompatible with the fundamental geometric methodology used by the historical craftsman, therefore it is hardly surprising that drawings were not used at all.

It must be emphasized that the primary motivation for modern manufacturing practice is economic, viz. automation and standardization. Out of necessity, assembly-line manufacturing demands that the individual identity of a particular specimen of the product is lost, i.e. all products are as similar as possible. Therefore, a much greater emphasis must be placed on an acceptable tolerance for inter-product uniformity at the expense of product self-consistency. This manufacturing focus is further exacerbated by standardization and out-sourcing which requires component parts to be completely interchangeable. The weak aspect of the modern method will always lie in the process of locating points with respect to a reference standard, even when the most sophisticated and accurate measuring devices are used to transfer the dimensions onto the construction grid. It is the intrinsic dimensional comparison between object and reference that cannot be controlled adequately[10] to maintain accurate dimensional relationships between points on the object.

Musical instruments gain no functional advantage from the inter-product consistency which is so essential to the modern approach, because only internal self-consistency is really critical to their correct functioning and assembly. Even when semi-automated manufacturing had begun to be used, for example in the Broadwood factory which produced over 3000 pianos per year in 1836,[11] or the Graf factory which employed in the 1830s over 40 workers operating on

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[8] J. Blüthner and H. Gretschel. *Lehrbuch des Pianofortebaues in seiner Geschichte, Theorie und Technik*. Weimar, 1886 (second edition). Chap. 16: Der Bau des Flügels.

[9] Hubbard, op. cit., pp.209-210.

[10] For some modern piano manufacturers, inaccurate point location has become such a serious manufacturing quality control issue, that they have been forced to turn to computer-aided manufacturing technology.

[11] D. Wainwright. *Broadwood By Appointment – A History.* Quiller Press, London, 1982, p.135.

specialized tasks in a work-station-based approach,[12] historical makers avoided the self-consistency problem by identifying lot numbers on component parts, ensuring that parts with the same numbers were directed to the same piano so that a proper fit was guaranteed. Minimal differences in size between instruments of the same type are of no consequence provided they are proportionally similar and correct relationships are maintained between the component dimensions on each individual instrument.

The German folk-saying *messen heißt vergleichen* tells us that to measure is to compare. Modern design methodology has been characterized as a comparison of dimensions between the object and a fixed reference; in contrast, the essential aspect of historical design is the derivation of new dimensions in an object by comparison to other dimensions in the object itself, the absolute size remaining arbitrary until a single generating modular dimension is fixed. This construction of dimensions within the object can be accomplished either indirectly, by physically marking out dimensions using dividers and (or) a modular scale, or directly, using geometrical constructions. Both of these methods can be observed in the stringed keyboard instruments of historical makers, the specifics of how they were applied being one of the important aspects which defined different regional schools of building.

THE CONNECTION BETWEEN ARCHITECTURE AND THE CONSTRUCTIVE CRAFTS

Practically-oriented architectural sources provide valuable information about the methods of historical artificers in general, since architectural design principles were considered to be the conceptual basis for design practice in all the individual crafts. John Dee, a leading Elizabethan mathematician, went so far as to say that 'the architect is not an Artificer himself, but the Hed, the Provost, the Director, and Judge of all Artificiall [i.e. man-made] workes, and all Artificers.'[13] As early as the Twelfth Century, the Spanish translator and philosopher Dominicus Gundissalinus considered that 'these many kinds of craftsmen are distinguished according to the different materials in which and out of which they work.' [14] In other words, the differences between the crafts were considered to lie primarily in working medium, and not in the design methods *per se*. The medieval master mason Mathes Roriczer concurs with this view, explaining that application depends on the rules and requirements of the specific craft, which is distinguished from others by its individual 'materials, forms, and measures.' [15] In the Italian Renaissance no specific training was available for an architect,[16] and the title was simply awarded to a master craftsman as a result of receiving his first building commission, often at a comparatively late age after years of work in the arts as painter, sculptor, or occasionally in the building trades (e.g. Palladio). Henry Wotton described the basis for the common working principles and practical kinship amongst the crafts: 'Wherof the first sort, howsoever usually set down by Architects as a piece of their Profession: yet are in truth borrowed, from other Learnings there being between Arts and Sciences, as well as betweene Men, a kinde of good fellowship, and communication of their Principles.' [17]

These connections with architectural design practice are very clearly seen in the craft of cabinet-making. Thomas Chippendale's well-known design book *The Gentleman and Cabinet-*

[12] D. Wythe. *Conrad Graf (1752-1851). Imperial Royal Court Fortepiano Maker in Vienna*, Ph.D. Thesis, New York Univ., 1990.
[13] Preface to Billingsley's 1570 English edition of Euclid's elements. Quoted in: R. Wittkower. *Palladio and Palladianism.* George Braziller, New York, 1974, p.99.
[14] Dominicus Gundissalinus. De divisione philosophiae. Quoted in: L. Shelby, The geometrical knowledge of the mediaeval master masons. Speculum 47 (1972) 395-421, p.403.
[15] Shelby, 1972, op. cit., p.416.
[16] J. Ackerman. Architectural practice in the Italian Renaissance. J. Soc. Architectural Historians XIII (1954) 3-11.
[17] Henry Wotton, op. cit.

maker's Director,[18] published to illustrate the firm's furniture designs for potential customers, provides a great deal of information about the modular geometry on which the designs were based. The book begins with an exposition of the five orders of Vitruvian architecture, and stresses their importance to the cabinet-maker. It is worthwhile to quote from Chippendale at length:

> 'Without an Acquaintance with [architecture] ... the Cabinet-Maker cannot make the Designs of his Work intelligible, nor shew, in a little Compass, the whole Conduct and Effect of the Piece. These, therefore, ought to be carefully studied by every one who would excel in this Branch, since they are the very Soul and basis of his Art. Of all the Arts which are either improved or ornamented by Architecture, that of Cabinet-Making is not only the most useful and ornamental, but capable of receiving as great an Assistance from it as any whatever.'

The extrapolation of Chippendale's remarks from cabinet-making to the design of stringed keyboard instruments is a logical and reasonable step. Even in terms of the decorative aspects of an instrument, the influence of architectural practice is quite evident, for instance in Pascal Taskin's description of a harpsichord as being painted 'under the lid in the orders of architecture.'[19] Just as Chippendale's furniture designs are highly architectural, and reflect the prevailing fashion, the more expensive stringed keyboard instruments were often decorated according to this style. This influence can be seen, for example, in the Doric column legs in vogue for Viennese pianos in the 1820s, or the Biedermeier designs of the 1830s. In this respect a wealthy customer would have regarded instruments and household furniture in the same context, and selected a design which harmonized with the interior and exterior architectural scheme of his house.[20]

While craftsmen made use of mathematics, especially geometry, in their practical work, they also made a valuable contribution to the development of mathematics and science. The sixteenth-century Dutch mathematician Rudolf Snellius recognized this in his study of the work of merchants, blacksmiths, and musicians, and attempted to learn from their unique approach to problem solving.[21] The abilities of craftsmen were shown not only by their geometrical manipulation, but also in their skills in empirical technology. 'Artisans ... having outgrown the constraints of guild tradition and being stimulated to inventions by economic competition ... [were] the real pioneers of empirical observation, experimentation, and causal research.'[22] Makers of musical instruments were included in this elite group of technologically advanced artisans, yet their efforts generally went unrecognized and unrecorded, because there was no incentive to 'systematize their discoveries and raise them from the level of rules of thumb to exact scientific laws.'[23]

Henry Wotton's 'Commodie, Firmenes and Delight' imply that a craftsman's primary concerns were directed toward producing, as efficiently as possible, a high quality product which was both functionally and aesthetically correct. Functionality requires that a musical instrument must work properly mechanically and acoustically, to perform its role as a tool for music making: keyboard and action must fit and work correctly in relation to the strings; the stringband with its projected string lengths must fit the case properly; plucking or strike points must be as required (desired); there must be room for placing the bridge an adequate distance from the edge of the

[18] Thomas Chippendale. *The Gentleman and Cabinet-maker's Director*. Reprint of the Third Edition of 1762, Dover, New York, 1966 (Preface).

[19] E. Closson. Pascal Taskin, Sammelbaende der Internationalen Musikgesellschaft, 1910-1911, p.234. Reprinted in Hubbard, op. cit., p.217.

[20] N. Harris. *Chippendale.* Chartwell Books, Syracuse, NJ, 1989.

[21] K. van Berkel. A note on Rudolf Snellius and the early history of mathematics in Leiden. In: C. Hay, Ed. Mathematics from Manuscript to Print 1300-1600. Oxford Univ. Press, 1988, pp.156-161.

[22] E. Zilsel. The sociological roots of science. Amer. J. of Sociology 47 (1941/2) 550-551.

[23] H. Cohen. Beats and the origins of early modern science. In: V. Coelhe, Ed. *Music and Science in the Age of Galileo.* Kluwer, Dordrecht, 1992, pp.5-44 (p.27).

instrument, and so on. The complexity of the interactions between these various functional requirements suggests that the only practical way of proceeding was to propagate the design, in particular the basic dimensions and geometry, directly in terms of the location and size of the main functional components. This conclusion will be useful in applying basic historical working principles to the practice of making keyboard instruments.

THE PRACTICAL GEOMETRY OF THE CRAFTSMAN

John Dee was keenly interested in the practical applications of mathematics and called architecture an 'Ars mathematical.' [24] An eighteenth-century builders' dictionary defines architecture as: 'A Mathematical Science, which teaches the Art of Building, a Skill obtain'd by the Precepts of Geometry, by which it gives the Rules for designing and raising all Sorts of Structures, according to Geometry and Proportion.' [25] That architecture and practical geometry were generally regarded as almost synonymous is made clear from the statements to this effect in so many historical treatises.

The architect Sebastiano Serlio declared geometry to be indispensable to the practice of any art: 'How needfull and necessary the most secret Art of Geometrie is for every Artificer and Workeman ... Geometrie is the first degree of all good Art.' [26] Dominicus Gundissalinus described an 'artificer of practice [as] he who uses [geometry] in working ... Craftsmen are those who exert themselves by working in the constructive or mechanical arts – such as the carpenter in wood, the smith in iron, the mason in clay and stones, and likewise every artificer of the mechanical arts – according to practical geometry.' [27] Skill in practical geometry is invariably cited by historical sources as indispensable, and fundamental, to the practice of any craft. The thirteenth-century French mason Villard de Honnecourt described the crucial role of geometrical forms in the crafts of masonrie and carpenterie.[28] 'There is no artifice nor handicraft that is wrought by man's hand but it is wrought by geometry,' wrote an anonymous French source in about 1400, concluding that 'men live all by geometry.' [29] Mathes Roriczer stated that 'every art ... arises out of the fundamental basis of geometry.' [30] Gundissalinus continued his observations on the practical use of geometry by craftsmen in very specific terms, saying that 'each [craftsman] indeed forms lines, surfaces, squares, circles, etc. in material bodies in the manner appropriate to his art ... The office of practical geometry ... in the matter of fabricating is to set the prescribed lines, surfaces, figures, and magnitudes according to which that type of work is determined.' Thus it can be confidently concluded that geometric methods provided the common design methodology and links between all the crafts. Precisely how this was utilized in design and construction by individual artificers was determined by the specific requirements and rules of his craft.

Modern 'school' mathematics is quite different from the mathematics of the historical craftsman, however this divergence is comparatively recent. The pragmatic approach of utilitarian mathematics is illustrated for instance by the commercial arithmetic used by Nicolas Chuquet: 'This style of working, without any proofs, these statements of rules, quite different from those which we have today since they indicate a procedure to follow and not formal reasoning, is in keeping with the habits of the time [1484].' [31] The masons, carpenters, and other trades used

[24] R. Wittkower, 1974, op. cit., p.98.
[25] *The City and Country Purchaser's and Builder's Dictionary.* Third Edition, Richard Neve, London, 1736.
[26] Sebastiano Serlio. *The Book of Architecture.* Edition transl. by Robert Peake, London, 1611. Modern reprint: A. Santianello, Ed. Benjamin Blom, New York, 1970. First Book, First Chapter, p.1.
[27] Shelby, 1972, op. cit., p.403.
[28] Ibid., p.395.
[29] Ibid., p.396. This source is contemporary with Henri-Arnaut de Zwolle.
[30] Shelby, 1972, op. cit., p.417.
[31] P. Benoit. The commercial arithmetic of Nicolas Chuquet. In: C. Hay, Ed. *Mathematics From Manuscript to Print 1300-1600.* Oxford University Press, 1988, pp.97-116 (p 110).

geometry in much the same manner as Nicolas Chuquet's arithmetic served commercial applications. Even today in cultures lacking formalized mathematical training, there are living oral traditions of this kind, for instance the intuitive 'street' mathematics used by some tradesmen and vendors in their daily life.[32] This utilitarian approach to problem solving is quite alien to the 'correctness' of formal mathematics, yet the methods are demonstrably effective and simple to apply. To assess the knowledge and skills that the instrument building craftsmen can be expected to have had, and explore the origins of this approach to problem solving, a brief excursion through the development of medieval geometry is useful.

At this point an important caveat is necessary. It is tempting to suppose that, given the current state of modern mathematical knowledge, it should be relatively simple to determine how the historical craftsman used geometry. In particular, it is often assumed that the various historical sources describing practical geometry ought to provide the craftsman's methods,[33] despite the fact that these pragmatic shop techniques were rarely recorded, and certainly never systematically. This false conclusion is based on an incorrect division of mathematics into two mutually exclusive theoretical and practical divisions and the implied association of the craftsman's methods with the content of published treatises on practical mathematics.

The medieval view of mathematics certainly placed a strong distinction between the theoretical and the practical. In terms of geometry, Hugh of St. Victor wrote in his short twelfth-century treatise on practical geometry that 'the entire discipline of geometry is either theoretical, that is speculative, or practical, that is active. The theoretical is that which investigates spaces and distances of rational dimensions only by speculative reasoning; the practical is that which is done by means of certain instruments, and which makes judgements by proportionally joining together one thing with another.' [34] The study of 'theoretical' geometry, based on Euclid's *Elements*, was confined to learned society and had very little connection with practical geometry, which involved mainly techniques for surveying and metrology. Dominicus Gundissalinus extended Hugh's classification of practical geometry, including for the first time the important *geometria fabrorum*, or practical geometry of the craftsman, along with the traditional surveying and mensuration.

Attempts to develop practical geometry in a theoretical context, with propositions and proofs, were generally ignored, not the least because the users of practical geometry – the masons, surveyors, carpenters, and so on – were unlikely to be able to read Latin texts. These include the *Practica geometriae* of Leonardo Pisano (Fibonacci), published in 1220, which specifically differentiates the target audience into the scholars, and 'those who would proceed by common usage or, as it were, lay custom.' [35] When vernacular works directed toward manual workers began to appear, these were generally ignored by scholars.[36] Thus the dichotomy between the practical mathematics of the artisans and the theoretical mathematics of the scholars was further fuelled by social circumstances. More important than this consideration, though, is to recognize that a craftsman would have seen no need at all to justify mathematically the procedures he used daily, nor would he have seen any need to document them systematically. These were geometrical constructions that clearly worked well, and provided an effective technical basis for design and construction of such massive undertakings as medieval cathedrals. This pragmatic geometry of the craftsman was fundamentally distinct from both theoretical Euclidean geometry, and the more formal practical geometry of the scholar, therefore it should properly be recognized as a third and distinct type of geometry – we call it constructive geometry.[37]

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[32] T. Nunes, A. Schliemann, A. Dias, and D. Carraher. *Street Mathematics and School Mathematics.* Cambridge University Press, 1993.

[33] Koster, op. cit., even suggests that the omission of a particular technique from a particular compilation of practical geometry is an indication that the technique was not used by craftsmen.

[34] Shelby, 1972, op. cit., p.401.

[35] S. Victor. *Practical Geometry in the High Middle Ages.* Amer. Phil. Soc., Philadelphia, 1979, p.47.

[36] Cohen, op. cit., p.28.

[37] Shelby, 1972, op. cit. The author identifies the distinct nature of the craftsman's non-mathematical geometry, using this term 'constructive geometry', to distinguish it from theoretical and practical geometry.

The techniques described in the *Sketchbook* of Villard de Honnecourt reveal that the author's knowledge of formal geometry was actually quite weak.[38] Similarly, some of the constructions in the printed works of the late mediaeval German masons Mathes Roriczer, Lorenz Lechler, and the goldsmith Hanns Schmuttermayer[39] – basically handbooks intended as teaching aids for the oral training of apprentices – can be shown to be theoretically incorrect. The presentation of the material in these pamphlets is quite haphazard, an organization which reflects the oral learning tradition in which these authors worked. Comparison with scholarly contemporary works on practical geometry shows just how a mason could have followed a prescribed construction sequence without the slightest knowledge of the formal Euclidean geometry on which it was based. The techniques of the master mason were step-by-step procedures (sometimes with hundreds of steps) which involved only the physical manipulation of geometrical shapes – triangles, squares, circles – using the available tools, and no mathematical calculation at all. In fact calculation was specifically avoided by the skilled manipulation of the masons' tools. Geometrical constructions in effect were used as a mechanical calculator, thereby circumventing any need to manipulate numerical quantities, a theme which constantly reappears in historical sources on applicable mathematics.[40]

Reconstructing the geometry used by the historical craftsman is not as simple as might be supposed. Continuity of oral traditions in the use of constructive geometry has been broken for several generations and these skills have essentially been lost from the modern technical vocabulary.[41] Trends in the development of scholarly mathematics have continued to distance it from the pragmatic methods of the historical craftsman. The increasingly digitally-oriented modern technological world is not compatible with the visual or analogue nature of constructive geometry, so it is likely to be further marginalized by computer technology.[42] Geometry taught in a modern education, even that of mathematicians, strongly emphasizes the algebraic or coordinate approach at the expense of spatially-oriented methods. Finally, as we have seen, scholarly mathematical sources, including those devoted to practical geometry, are generally misleading, because they are indicative of neither the knowledge nor the practical skills of a contemporary historical craftsman.

TOOLS FOR CONSTRUCTIVE GEOMETRY

According to the fourteenth-century *Holkham Picture Book*, God, as architect of the universe, is supposed to have created the world with a giant pair of compasses.[43] Compasses (also called dividers), straightedge, and set squares were the layout tools of the craftsman. Their supreme importance is described in many historical sources, and can be deduced from iconographical

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[38] Ibid., pp.406-407.

[39] Mathes Roriczer. *Geometria Deutsch* and *Büchlein von der Fialen Gerechtigkeit*. Regensburg, 1486 and 1488. Fascimile Edition, Ferdinand Geldner, Wiesbaden, 1965; Hanns Schmuttermayer. *Fialenbüchlein*. *ca*1486; Lorenz Lechler. *Unterweisung*, 1516. The three booklets of Roriczer and Schmuttermayer are reprinted and translated in Shelby, 1977, op. cit. Lechler's booklet is discussed in: Shelby, 1972, op. cit., pp.419-420.

[40] Captaine Thomas Rudd (deceased), Chiefe Engineer to his late Majestie, London*. Practical Geometry in 2 Parts: The second containing a hundred geometrical questions with their solutions and demonstrations, some of them being performed arithmetically and others geometrically, yet all without the help of algebra.* Imprinted at London by J.G. for Robert Boydell and to be sold at his shop in the Bulwarke near the Tower, 1650.

[41] There are some exceptions where artisinal traditions have (partially) been continued, for instance in modern luthiery.

[42] However, it should be noted that the principles on which many computer-aided design programs are based, defining relationships between entities rather than absolute coordinates, have something in common with the historical craftsman's geometry.

[43] N. Coldstream. *Medieval Craftsmen: Masons and Sculptors.* Univ. of Toronto Press, Toronto, p.5. Many additional examples of iconographical evidence are reproduced, showing the importance of layout tools.

evidence as well. For example, the coats of arms of both the Masons' and Carpenters' Guilds in the City of London always prominently placed a display of pairs of compasses as the central image.[44] Master masons are frequently portrayed in paintings holding these layout tools, or placed beside them in their burial imagery. In his *Fialenbüchlein*, a late fifteenth-century booklet on the layout procedures for pinnacles, Hanns Schmuttermayer made it clear that 'from the beginning the [high art of construction] has its basis in the level, set-square, triangle, compass, and straightedge.'[45] Serlio's frontispiece shows the tools he expected to be used for practical geometric constructions – the square, compass and straightedge.[46]

Layout tools were sized in relation to the artifact being constructed. The square is, of course, indispensable for constructing short perpendiculars, but very large squares (similar to the large modern carpenter's square) would probably not have been used to construct long perpendiculars, as these are most accurately and simply produced with a direct geometric construction.[47] Regular compasses were available in different sizes, but were probably not practical for transferring dimensions longer than about 50 cm, for which the beam compass (trammel) was used. The historical trammel consisted of a piece of wood of appropriate length, on which two points were slid and held by a simple wedge to define a dimension (Figure 1). Roubo[48] illustrates the architectural carpenters' workshop, where enormous doors and panelling are being made. In the corner of the workshop a huge trammel is clearly visible leaning against the wall, its size commensurate with the size of the objects being laid out in that workshop. A trammel of length about 1.5 m is appropriate for laying out long dimensions for keyboard instruments such as harpsichords or pianos, but large compasses may have sufficed for the smaller instruments such as virginals, clavichords or square pianos. In his monumental work on organ building, Dom Bedos confirms the list of simple tools required for instrument layout. He specifies to obtain 'a pair of compasses; a large beam compass, or trammel; squares of several sizes, one very large; rules of all lengths; and dividers of several sizes, with legs from 1 pied 6 pouces to 5 or 6 pouces long, the latter to be sturdy, the larger ones like stonecutters' or carpenters' dividers, and the smaller ones like joiners'.'[49] He concludes that it is not necessary to illustrate these tools – in particular the layout tools – 'as they are familiar enough.' Inventories taken in the shops of various harpsichord builders also confirm the prominence of the compasses and trammel.[50]

It is interesting to consider why craftsman should have limited themselves to these particular simple layout tools. Scholarly geometers approached Euclidean constructions as an intellectual exercise, accepting the limitations imposed by straightedge and compass as a theoretical necessity for their arguments. However, even though there was no such intellectual requirement for the constructive geometry of the historical artificer, all useful geometric constructions can be performed easily with the basic layout tools, so no practical benefit would have been achieved with more sophisticated tools of mensuration. The obvious focus of historical craftsmen on simple layout tools is a strong indication that their working practice was based on constructive geometry.

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[44] W. Hazlitt. *The Livery Companies of the City of London.* Benjamin Blom, New York, p.405 and p.564. In the City of London, makers of musical instruments were organized as a part of the Carpenters Guild.

[45] Shelby, 1972, op. cit., p.418.

[46] Serlio, op. cit.

[47] For instance, the transverse perpendicular lines required in harpsichord or piano layout, and the line defining the front edge of the instrument.

[48] Roubo. *L'Art du Menuisier, 1769.* Plate XI - Vue interieure de la boutique d'un menuisier.

[49] Dom Bedos. *L'Art du Facteur d'Orgues*, 1776.

[50] Hubbard, op. cit. Many inventories are listed in detail in the appendices.

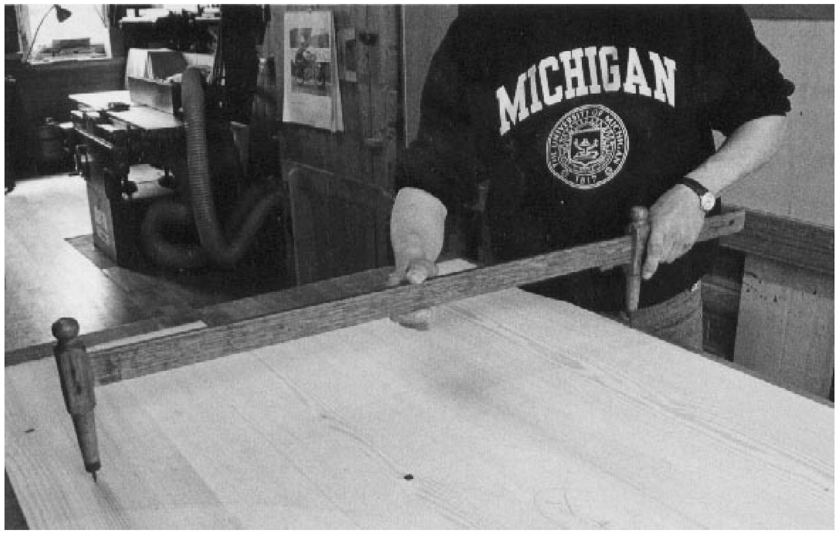

Figure 1. *Use of an historical trammel for transferring long dimensions in geometric constructions.*

The proportional dividers are a useful device which was available to the historical craftsman to multiply or divide a dimension by some constant without the need for numerical calculation. This simple device consists of two legs, marked out by some geometric method to provide the desired proportions, and connected together at a point generally located towards one end. By rotating the legs about the pivot so that the reference dimension on the tool matches the required measurement, proportional dimensions can be derived from it by setting a pair of compasses between the appropriate marks.[51] Galileo described effusively how this tool circumvented the need for formal mathematical training and served as a mechanical calculator: 'Everything that one would otherwise have to learn through long, arduous study of geometry and arithmetic ... can be mastered in a matter of days with my proportional dividers.' [52] Nevertheless, although the historical instrument maker probably made use of proportional dividers where calculation could not be avoided, for example when determining the fret positions on clavichords or lutes, such calculations are not necessary when laying out the basic geometry of a stringed keyboard instrument.

THE ROLE OF PROPORTION AND HOLISM

Leone Battista Alberti defined beauty as 'the harmony and concord of all the parts achieved in such a manner that nothing could be added or taken away or altered except for the worse.' [53] This succinctly describes the historical craftsman's holistic view of his design, the main goal of which was to achieve a correct and satisfying relationship between the parts and the whole, i.e. a harmonious balance between reductionism and holism.[54] This theme is alluded to in some way in most historical architectural works. Henry Wotton, for example, began his architectural pamphlet

[51] For details see: R. Gug. Le compas de proportion ou un instrument de mathematiques au service de la facture instrumentale ancienne. Musique Ancienne 20 (December, 1985) 3-23.
[52] Le Opere di Galileo Galilei, edizione Nazionale, Vol. 2, p.369. Quoted in R. Gug, Ibid. 'via veramente regia, la quale con l'aiuto di questò mio Compasso in pochissimi giorni insegna tutto quello, chedalla geometrica e dall'aritmetica, per l'uso civile e militare, non senza lunghissimi studii per le vie ordinarie si riceve' (trans. the authors).
[53] Leone Battista Alberti. *Ten Books on Architecture*. Florence, 1485. Transl. J. Leoni. London, 1755. Modern edition: J. Rykwert, Ed. Tiranti, London, 1965. Book 6, Chapter 2.

with: 'The Precepts thereunto belonging, doe either concerne the Totall Posture (as I may tearme it), or the Placing of the Parts.' [55] In his *Four Books of Architecture*, Palladio declared that 'beauty will result from the form and correspondence of the whole, with respect to the several parts, of the parts with regard to each other, and of these again to the whole; that the structure may appear an entire and compleat body, wherein each member agrees with the other, and all necessary to compose what you intend to form.' [56]

Instrument makers, too, concerned themselves with the problem of defining and implementing correct proportions. That this was regarded to be an important aspect in the construction of stringed keyboard instruments is illustrated, for instance, by the 1557 rules of the Guild of St. Luke in Antwerpen. These regulations state that a prospective builder who wished to join the Guild must submit a 'test piece ... harpsichord, five feet long or thereabouts, or longer should he so desire, well and truly wrought, in the correct shape and proportions.' [57] Thus correct proportions are a crucial element for success, while relative freedom is granted for the absolute size (length).

The practical basis for achieving consistently correct proportional relationships in design relied on a very simple and well-documented concept. A single dimension – called the module – was selected, and formed the basis for deriving all subsequent dimensions. 'With but a single basic dimension given, the Gothic architect developed all other magnitudes ... by strictly geometrical means.' [58] This modular approach was ubiquitous in historical design, although the varied geometrical techniques by which new dimensions were derived from the module tend to obscure this fact. The sixteenth-century German master masons gave specific instructions how this was to be done, identifying the modular dimensions by descriptive metaphors based on common language. For example, Schmuttermayer defined a diminishing sequence of eight modular dimensions, beginning with a square of any convenient size – the side of this is the generating module, which he called the *alt schuch* (old shoe). From this he constructed seven *new schuch* (new shoe) dimensions by inscribing a square inside the previous one at the midpoints of its sides[59] (Figure 2). Successive shoes, in Schmuttermayer's modular scheme, are therefore related in the ratio $\sqrt{2}/2$. Lechler's scheme was constructed similarly, with fewer inscribed squares, and some dimensions derived by other constructions.[60] He also used the old and new shoe metaphors for the modular dimensions, as well as various other terms such as old and new mullion, template, and measurement.[61]

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[54] This approach contrasts with modern design methodology which tends to favour the former at the expense of the latter.
[55] Henry Wotton, op. cit.
[56] Andrea Palladio. *The Four Books of Architecture*. Venice, 1570. Transl. Isaac Ware. London, 1737. Modern reprint Dover Publications, New York, l965. First Book, Chapter 1.
[57] G. O'Brien. *A Ruckers Harpsichord and Virginal Building Tradition*. Cambridge Univ. Press, New York, 1990, pp.300-301.
[58] O. von Simson. The gothic cathedral: design and meaning. J. Soc. Architectural Historians XI (1952) 6-16.
[59] Shelby, 1977, op. cit., p.128.
[60] Shelby, 1972, op. cit., p.419.
[61] Shelby, 1977, op. cit., p.183. This illustrates that the terms were merely meant descriptively, and not, as suggested in: Koster, 1998, op. cit., p.3, with some implied connection to a local foot unit of measure. According to Shelby, 1977, p.190, the medieval German *schuch* means shoe or measure, in the generic, rather than specific, sense. There is an adequate amount of evidence to support the arbitrary size of the generating module in architectural design, a decision which could be adapted to the requirements of any particular building.

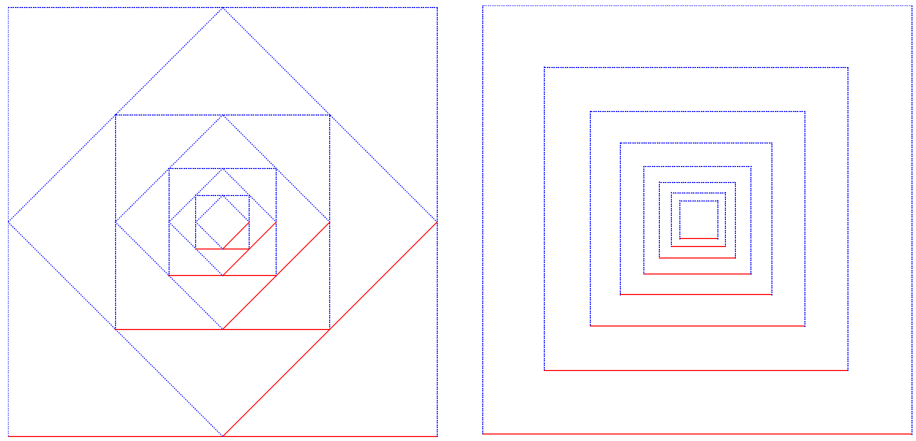

Figure 2. *Schmuttermayer's geometric derivation of new schuch dimensions from a generating module (alt schuch), the side of the largest square. Each dimension is related to the previous one by the ratio $\sqrt{2}/2$, since it is derived from it as the diagonal of half a square. The lower figure illustrates the relative sizes of the sequence of schuch dimensions.*

Vitruvius wrote in the Ten Books of Architecture that 'one of these parts will be the module; and this module once fixed, all the parts of the work are adjusted by means of calculations based upon it.'[62] The classical ideals of Vitruvian architecture were revived and described at great length in Renaissance treatises which demanded 'that inside as well as outside ... [buildings] have the module as their common denominator.'[63] While the gothic architect had some freedom in the specific choice of module, classical design principles demanded a specific relationship between a building and its module. 'Following Vitruvius, Renaissance architects accepted the diameter of the column, the module, as the standard unit of measurement, and by multiplying and dividing it, they welded details as well as whole buildings into metrically related units.'[64] The English architect Inigo Jones explicitly revealed his geometric thinking in the extensive annotations he marked into his personal copy of Palladio. Jones determined modular scales for Palladio's designs and, for many of the diagrams, clearly pricked these into the paper of the book with dividers. Inked lines show how Jones analysed the derivation of subsequent dimensions from the module.

Lechler gave a specific example of a convenient dimension that could be used for the generating module, 'the wall thickness of the choir, whether it be large or small.' [65] This illustrates the general principle that the modular starting point was based on convenience or practicality – for instance, in a large architectural project completed over many decades, the modular unit provided the continuity between successive generations of workers. Even the universal Renaissance module, the column diameter, was still an adaptable unit that could be made unique to any particular building in order to meet practical necessity and convenience. This sort of restrictive freedom may seem strange to the modern designer. The module could certainly not have had an *a priori* basis in any local unit of measurement, 'but only on a unit of measurement which was adaptable to each individual building and peculiar to it. The necessity for this needs no lengthy comment ... An absolute metrical relationship of part to part, and of parts to the whole [in the human body] can only be expressed by postulating a standard of

[62] Vitruvius, Book IV, Chapter 4. Transl. M.H. Morgan. Dover, New York, 1960, p.110.
[63] R. Wittkower, 1974, op. cit., p.59.
[64] Ibid., p.59.
[65] Shelby, 1972, op. cit., p.419.

measurement, e.g. the head or the face, as was done by Vitruvius and his Renaissance followers.'[66]

The modular scales shown on most of the furniture designs in Chippendale's *Director* [67] provide evidence for the use of modular dimensions by cabinetmakers. These scales may seem superficially similar to those of modern engineering drawings, but examination reveals that they cannot have been based on any fixed relationship with the local unit (for instance as a modern one inch per foot scale). The modules in different drawings are different in absolute terms and clearly relate to important dimensions in the piece being represented, such as the height of a cabinet or bookcase. Chippendale explains how the absolute size of many of the pieces can be adjusted to fit a particular customer's requirements, by changing the size of the module with respect to its representation in inches or feet. This confirms again that practical considerations and convenience governed the choice of module in relation to specific requirements, rather than some inflexible standard. Some of the diagrams show large size moulding profiles, illustrating how even these were all to be derived from the basic modular unit:

> 'Take the whole Height of the upper Part of the Bookcase, and divide it into twenty equal Parts; one of which divide again into three equal parts one way, and into four the other; then divide one of those three Parts into twelve equal Parts: then draw a Diagonal from Corner to Corner, and in one of the Divisions to take off Quarters, Halves, and three Quarters. The Mouldings are drawn from this Scale.'[68]

Thus the height of the bookcase in its modular representation is divided into 60 parts and the diagonal line is used to obtain the quarters of those parts for small dimensions, such as those used to define the geometry of the moulding profiles.

Although the basic modular principle is clear enough in the drawings of the *Director*, it is highly unlikely that a cabinetmaker would have used them as construction plans, extracting dimensions with dividers, scaling these, and transferring them to the piece being made. The drawings were probably mainly intended to form a sort of sales catalogue, and not explicit aids for laying out purposes in the shop. Chippendale's omission of information about internal construction may be related to this objective; alternatively he may have wished to keep these aspects of the designs to himself, or, more likely, the expected practice was so commonly known amongst cabinetmakers that no explanation was necessary for that audience. Since the modular dimensions of a piece of furniture would have been marked out before the completion of the exterior and its decoration, the basic geometry of a design will coincide exactly with some internal dimensions in the structural framework, rather than the exterior of a piece. This conclusion also applies to stringed keyboard instrument designs, for which the outer case often obscures the geometric relationships used by the builder. Therefore, it is essential to determine the likely sequence of layout and construction for an artifact before observations can be properly related to its geometry.

THE PROBLEM WITH PROPORTIONS

Proportional relationships do not manifest themselves into an historical artifact by chance, whim, or mysticism. They are the explicit result of the geometric scheme which was used to derive dimensions from a common module. In practice this was accomplished with the layout tools, either directly, using geometric constructions to manipulate dimensions derived from the module, or indirectly, through modular measurement, i.e. by transferring multiples and divisions of the module from a modular scale. The difference between these two approaches is technical rather than conceptual – both are intrinsically constructive geometric procedures. In terms of instrument building shop practice, direct constructive geometry is more likely to have been associated with

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[66] Ibid., p.59.
[67] Chippendale, op. cit.
[68] Chippendale, op. cit., Plate XCIII.

designs conceived and built on the bottom, since the entire plan is visible in layout. In contrast to this, a design defined by a sequence of modular dimensions can be constructed using modular measurement, without any need to develop a full-scale plan on the bottom of the instrument. Direct constructive geometry can produce highly accurate proportional relationships and internal self-consistency within a particular instrument – this is the nature of the operation of a trammel and (or) compasses – while indirect modular measurement has a tendency to produce inconsequential inaccuracies. These observations have important consequences for the analysis of extant instruments.

A common approach in proportional analyses involves the reporting of essentially random observed relationships between dimensions. This method is fundamentally flawed from the outset because, at best, only an abstract connection between object and craftsman can ever be inferred from such mathematical exercises. Despite the often huge quantities of complicated data reported,[69] almost no useful knowledge can be gained about working methods and design. To make a meaningful connection with design technique requires the identification of only those very special proportional relationships, or dimensions, which were the direct consequence of a builder's actions. This can only be accomplished by identifying precisely which points the builder originally constructed, and identifying the geometric method he used to locate them.

It is always possible to find a great many abstract proportional relationships in an object[70] – regardless of whether it is biological, physical, or man-made – all the more so if even a moderate tolerance is permitted in numerical comparisons, as is so often the case in the published literature. 'Man has written infinitely about proportions. Every year books appear about new triangles, rectangles, polygons, golden or otherwise and keys and rules trying to solve the architectural secrets of the ancients,' [71] but these sorts of results can never be regarded as anything more than a 'study in geometrical shapes' rather than elucidating a craftsman's work.[72] That is to say, such proportional circuses are deterministically useless without the essential link to the mechanistic process which led to the extant form of the object, a critique which can be applied to most of the published proportional analyses of musical instruments.[73] By considering a reasonable library of potential proportional schemes, and allowing a moderate tolerance in numerical accuracy (often not even stated), the most marvellous constructions can be created on paper,[74] but these, in general, will have no connection with workshop practice or historical design principles. In any useful analysis of an extant artifact, proportions must be causally related to the pragmatic methods of the workshop which created it, and not remain as sterile attributes.

PROPORTIONS AND GEOMETRY

The numerically-focused modern person might expect that whole number, or fractional, ratios of dimensions are simpler to implement than irrational ratios, but it is important to keep an historical perspective on this. Simple shop constructions are well-known for generating dimensions in irrational proportions, for example: (i) the root proportional ratios $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, and so on (Figure 3); (ii) the golden ratio[75] $\phi = (\sqrt{5} - 1)/2$, about 0.618 (Figure 4); or (iii) the $\sqrt{\phi}$ ratio (Figure 5). Since no calculation, and no measurement, is required with these constructions, the

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[69] For example: K. Coates. *Geometry, Proportion and the Art of Lutherie*. Oxford University Press, 1985.
[70] M. Gardner. *Fads and Fallacies in the Name of Science*. Dover, New York, 1957. Chapter 15.
[71] E. Neufert. Bauordnungslehre, 1943. Quoted (footnote 19) in: M. Borissavlievitch. *The Golden Number and the Scientific Aesthetics of Architecture*. Tiranti, London, 1958.
[72] Borissavlievitch, op. cit., p.28.
[73] A notable exception which does discuss practicalities of construction is: R. Lundberg. In tune with the universe: The physics and metaphysics of Galileo's lute. In: V. Coelho, Ed. *Music and Science in the Age of Galileo*. Kluwer, 1992, pp.211-239.
[74] See for instance Coates, op. cit.
[75] Note than some authors use this symbol to denote the reciprocal of the number we have defined, i.e. $1/\phi$, about 1.618, and refer to this as the golden ratio.

incommensurability of the ratios is of no practical consequence, and the corresponding dimensions can be used just as easily as rational or whole number dimensions. For example, it has already been seen how Schmuttermayer's series of new shoes derived from the old shoe are related in terms of the $\sqrt{2}$ proportion.[76]

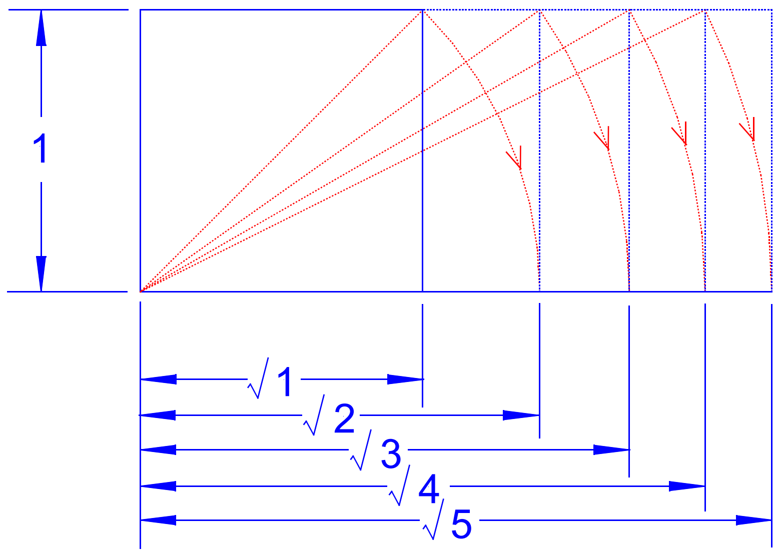

Figure 3. *Geometric construction of root proportional ratios from successive rectangles with the long side equal to the diagonal of the previous one. The initial 'rectangle' is a unit square.*

The golden ratio has captured the public imagination and become the central focus of a huge literature,[77] achieving almost a cult status in recent years.[78] The content of much of this material is quite fanciful. Claims to the ubiquitous appearance of the golden ratio in nature, man-made objects, buildings, art works, musical instruments,[79] and so on, are generally based on rather superficial analysis. 'Authors will draw golden rectangles that conveniently ignore parts of the object under consideration. In the absence of any clear criteria or standard methodology it is not surprising that they are able to detect the golden ratio.' [80] Perhaps as a reaction to the way in which the golden ratio has been popularized, it has become fashionable recently to deny that it

[76] Shelby, 1972, op. cit., p.417 and Fig 8.
[77] Some examples from this sprawling literature: M. Gylka. *The Geometry of Art and Life*, New York, 1946; Borissavlievitch, op. cit.; E. Huntley. *The Divine Proportion: A Study of Mathematical Beauty*. Dover, New York, 1970; L. Charpentier. *Die Geheimnisse der Kathedrale von Chartres*. Gaia Verlag, Koeln, 1986.
[78] G. Doczi. *The Power of Limits : Proportional Harmonies in Nature, Art and Architecture*. Shambhala Publications, Boulder, Colo., 1981.
[79] H. Heyde. *Musikinstrumentenbau.* Breitkopf und Härtel, Wiesbaden, 1986; H. Henkel. *Catalogue of Keyboard Instruments, Musikinstrumentensammlung Deutsches Museum, Munich*. Bochinsky, Frankfurt, 1994.
[80] G. Markowsky. Misconceptions about the golden ratio. College Math. J. 23 (1992) 2-9

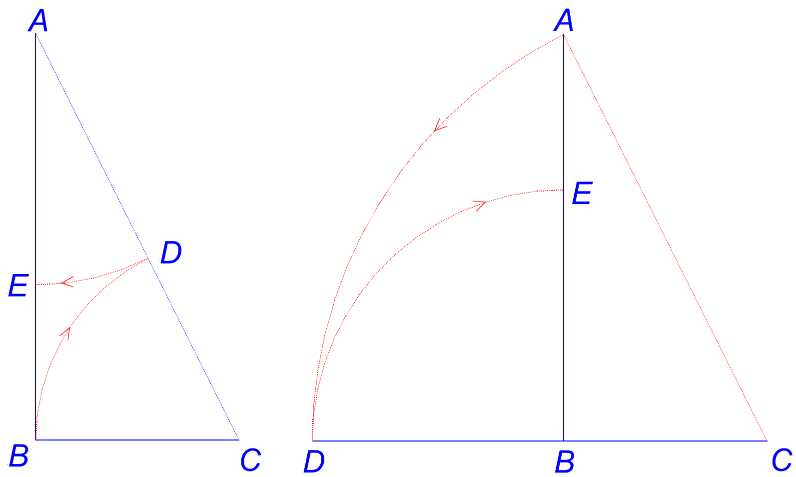

Figure 4. *Simple geometric (workshop) constructions for the golden ratio ϕ. The diagonal AC of a 1:2 rectangle (AB = 2BC) is used to construct point E dividing AB in the golden section. (a) (left) Intermediate point D is on the diagonal AC. The ratio EB:AE = AE:AB = ϕ. (b) (right) Intermediate point D is external to the rectangle. The ratio AE:EB = EB:AB = ϕ.*

ever played a role in historical design.[81] Negative assessments such as these reflect a lack of understanding of the extent to which the golden ratio occurs in so many practical shop constructions, especially those relating to rectangles generated from half a square (Figure 4), or the inter-locked golden rectangles that form a rectangle with sides in the ratio √5 to 1 (Figure 6). Indeed, it would be quite remarkable if historical craftsmen had actually been able to avoid the use of the golden ratio when using their practical constructive geometry. One only needs to consider the very beautiful geometric design basis of so many Viennese and South German pianos, applied over a period of almost a century from J.A. Stein to the mid-Nineteenth Century, to see the extent to which ϕ and √5 geometry could become the pervasive basis for an entire keyboard instrument building tradition.[82]

A commonly-cited argument against the historical use of the golden ratio is that these constructions do not appear in historical sources,[83] but this is specious for the reasons discussed above. In general, there is scant documentation of the pragmatic geometry – golden or otherwise – used by craftsmen. In fact, references to the golden ratio can actually be found in relevant historical sources, although they can easily be overlooked, because such sources tended to focus

[81] For example see: G. O'Brien. The use of simple geometry and the local unit of measurement in the design of Italian stringed keyboard instruments: An aid to attribution and to organological analysis. Galpin Soc. J. LII (1999) 108-171.

[82] S. Birkett and W. Jurgenson. Geometrical methods in stringed keyboard instrument design and construction. In: C. Chevallier & J. van Immerseel, Eds. *Matière et Musiques, The Cluny Encounter. Proceedings of the European Encounter on Instrument making and Restoration, Cluny, France, September, 1999.* Alamire, Pier, Belgium, pp.283-329. A comprehensive analysis of the geometry of Viennese and South German pianos is the subject of a forthcoming article.

[83] Koster, 1998, op. cit., p. 2. The same argument is repeated in: J. Koster. Cathedrals, cabinetmaking and clavichords Part I. Clavichord International 4 (2000) 6-12 (p 10), in which the author comments that 'Lechler does not mention these devices [i.e. the Fibonacci sequence or the golden ratio].' Given the very sketchy nature and incomplete content of Lechler's pamphlet, no meaningful conclusion can be drawn from the omission of a particular construction.

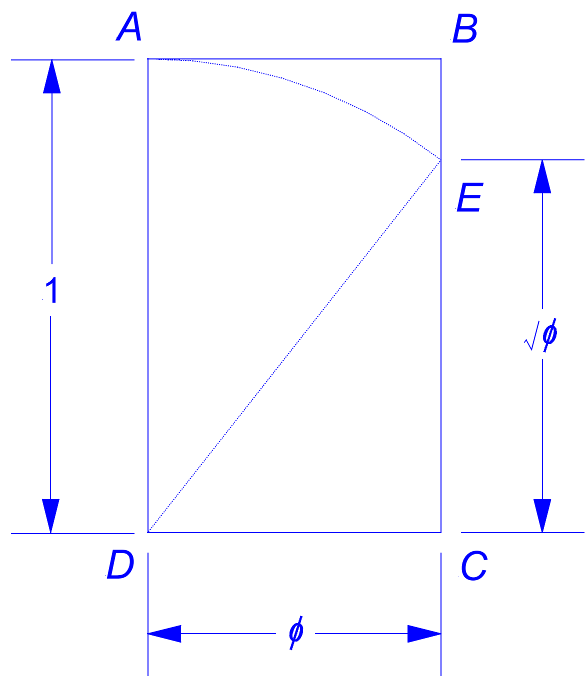

Figure 5. *Geometric (workshop) construction of the √φ ratio from a golden rectangle ABCD, with the sides in the ratio DC:AD = φ.. Point E is constructed by transferring the dimension of the opposite side as shown. The ratio EC:AD = √φ.. Triangle ECD has sides in the ratios φ : √φ :1, approximately 0.618:0.786:1.*

on the generated relationship, and the ratio was often not referred to by any special name.[84] For example, an English practical geometry published in 1650, intended as a handbook for surveyors, contains a method for constructing 'point D on the line BA, so that BD is to DA, as AD is to the whole line AB.' [85] This demonstrates that the fundamental property of the golden ratio was known to seventeenth-century English surveyors, and that they must have considered it useful enough to include its construction in their handbook.

The geometric properties of ϕ are utilitarian and can be easily applied in practical shop constructions with no understanding of the theoretical mathematics behind them. Practical constructions based on ϕ geometry had been known since Euclid, and earlier, even though the formal mathematical properties of ϕ were not systematized until Pacioli published his *Divina Proportione* in 1509. The golden ratio was well-known to Dürer, as confirmed by his personal notes, yet it does not appear in the published version of his *Painter's Manual* [86] of 1525, perhaps to show his personal respect and avoid overlap with Pacioli's recently published work. This omission was certainly intentional, because Dürer substituted Ptolemy's construction for the pentagon, while basing the other regular polygon constructions on Euclid. There can be little doubt that the simple geometric constructions involving ϕ were well known amongst craftsmen at

[84] The ratio is sometimes described in early sources as 'division in extreme and mean'.
[85] Captaine Thomas Rudd, op. cit. Question 50, pp 72-3. This is explicit evidence of a reference to practical construction of the golden ratio in 1650.
[86] A. Dürer, The Painter's Manual. Translated with commentary by W. Strauss. Abaris Books, New York. 1977.

this time. For instance, in his *First Book of Architecture*,[87] which was published in Paris in 1545, Serlio gives instructions for constructing a regular pentagon (reproduced in Figure 7). An intermediate step unequivocally shows the generation of the golden ratio[88] using the shop construction shown in Figure 4(b).

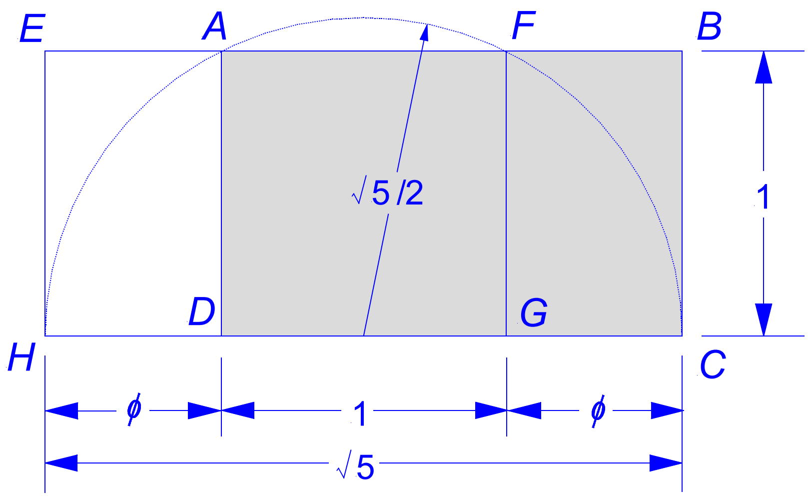

Figure 6. *Geometric (workshop) constructions involving inter-locked golden rectangles ABCD and EFGH, inside a rectangle with sides HC:BC in the ratio √5 :1. Points A and F are constructed where side EB intersects a semicircle with centre at the midpoint of side HC. AFGD is a square; FBCG and EADH are identical golden rectangles, with sides in the ratio FB:BC = EA:AD = ϕ.. The construction can be reversed, by attaching the square AFGD on the longer side of the golden rectangle FBCG, giving the larger shaded golden rectangle.*

The Renaissance revival of classical ideals generated a philosophically-motivated movement toward the use of rational proportions, especially those based on small whole number fractions, as a spatial expression of the Pythagorean 'harmony of the spheres.' [89] Dimensions related in such ratios are most easily produced by modular measurement, by transferring multiples and divisions of the module with dividers. Despite a preference for commensurable ratios, irrational proportions were also still permitted and used in this style of architecture, primarily the √2 proportion, which could be easily extracted from the diagonal of a square in the design, as described by Serlio: 'from one corner to the other, a line drawne to diagonus, and the length of the Diagonus shall bee the height of the flat.' [90] The height of a pedestal was very often derived from its width with a √2 construction.[91] Serlio documents many uses for irrational proportions: one of his specified room

[87] Serlio, op. cit. First Book, First Chapter, Folio 11.
[88] In: Koster, 2000, op. cit., p 10, the author states that, 'to my knowledge, [the golden ratio and Fibonacci sequence] are not described in any relevant source.' Even though Serlio is actually cited in: Koster, 1998, op. cit., p.2, the author has overlooked his construction of the golden ratio.
[89] R. Wittkower. *Architectural Principles in the Age of Humanism.* Tiranti, London, 1962, p.102.
[90] Serlio, op. cit. Fourth Book, Chapter 6, Folio 17.
[91] John Shute. *The First and Chief Groundes of Architecture.* London, 1563. Modern reprint. Gregg Press, London, 1965.

heights is in the proportion of √2 to the width of the room; by angling a set square he shows how to use the diagonal to determine conveniently the depth of column flutes; circles are constructed inscribed in squares which are inscribed in other circles; and, as mentioned above, he includes Euclid's pentagon construction based on the golden ratio. Palladio's comment 'that it will not be possible always to find this height in whole numbers,' [92] probably more restrained than most, is typical of the philosophical problems the Italian Renaissance architects had with the concept of incommensurable ratios. However, this had no bearing on their practical use.

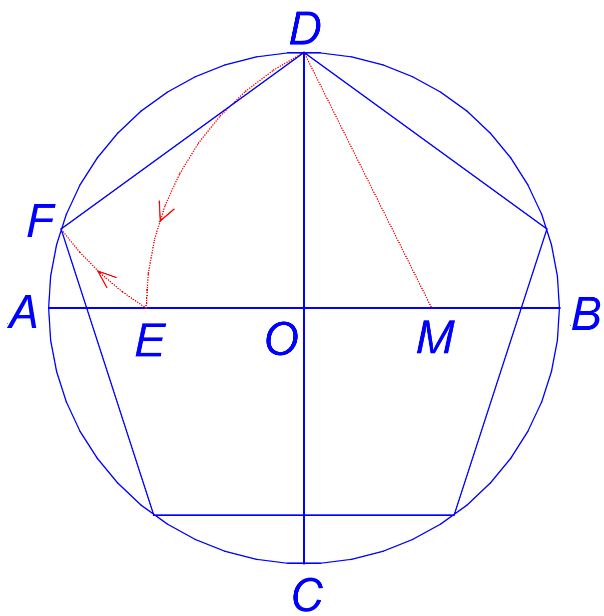

Figure 7. *Serlio's construction of a regular pentagon inscribed in a circle. The intermediate step constructs the golden section of radius OA at point E, using the method shown in Figure 4(b).*

Irrational ratios may also be suggested by modular measurement, using numerical sequences of whole number multiples of the module as approximations.[93] For example, the √2 ratio could be represented by a rational ratio of successive pairs of dimensions in one of the well-known sequences {5, 7, 10} or {12, 17, 24},[94] scaled according to the prevailing modular requirements of a design. Although Lechler's related modular dimensions were initially defined in terms of geometric construction from squares inscribed diagonally inside squares, in his written instructions for using these, he frequently approximates the √2 ratio by a rational fraction involving dimensions in the ratio 5 to 7.[95] The golden ratio is quickly approached by the ratios of successive pairs of numbers in the Fibonacci sequence {1, 2, 3, 5, 8, 13, 21 ...}, therefore craftsmen frequently represented ϕ by one of the rational ratios 3/5, 5/8, or 8/13. The use of Fibonacci numbers in a sequence of modular dimensions to define the geometry of a keyboard

[92] Palladio, op. cit. First Book, Chapter XXIII.
[93] Coldstream, op. cit.
[94] The last number is twice the first; the other number is one less than the middle between them. These dimensions can be observed in historical architecture. See: Coldstream, op. cit.
[95] Shelby, 1972, op. cit., p.419.

instrument was explicitly described as early as the Fifteenth Century by Arnaut de Zwolle.[96] Modular dimensions based on the Fibonacci sequence are particularly easy to remember, because of their self-generating properties, each one being the sum of the previous two, and provide a successful rule-of-thumb procedure that required neither theoretical background nor mathematical justification.

It is a moot point whether a craftsman made any distinction between an irrational ratio and its rational approximation, or simply viewed these as two representations of the same concept, which could be produced via alternative practical techniques. The ϕ critics who argue, on the basis of an inexact relationship, that the golden ratio is never really present in an artifact,[97] are creating an a-historical assessment criterion for analysing proportional relationships. The distinction is only important from a practical point of view, insofar as the implications for determining the geometric practice of a particular craftsman. The decision between irrational *vs.* rational dimensions really is of no consequence in a geometrically-based design practice. Incommensurable ratios arise naturally from a layout scheme that employs diagonal dimensions derived from an orthogonal module; commensurable ratios are a consequence of deriving orthogonal dimensions from an orthogonal module. Since the method used to construct these dimensions – direct geometric construction *vs.* modular measurement – does not alter their commensurability with respect to the module, this aspect of design cannot be associated *a priori* with a particular geometric working practice. However, it must be said that it is a natural procedure to transfer diagonal dimensions present in a design to the longitudinal direction with the trammel or compasses. Moreover, whole number multiples of the module are very easily constructed using modular measurement from a scale.

TWO APPROACHES TO GEOMETRIC DESIGN

Two conceptually distinct geometric approaches are possible for constructing the case geometry of a stringed keyboard instrument: (i) *inside-out*, i.e. construct the projected stringband and derive the case outline from it; or (ii) *outside-in*, i.e. construct the case outline in such a way that the projected stringband will automatically fit.

To apply the inside-out method, direct modular measurement may be used to locate the bridge, for instance by following a pre-determined scheme that gives the desired string lengths. The resultant bridge curve becomes the basis for the case geometry, with bentside curve and location defined by simple translation from the bridge curve (e.g. a specified distance from the bridge). The principle of this method is the same as that described by Arnaut in his instructions for laying out a clavisimbalum.[98] Inside-out geometry may be implied by instruments which exhibit a deeply-curved bentside and parallel bridge, such as Italian harpsichords, however some caution is required before reaching this conclusion, since such a case design may also have been the result of locating the bentside curve by direct measurement from a reference line at the front of the instrument, followed by inward translation to determine the bridge curve.

The alternative working practice, based on outside-in geometry, is the approach Hubbard described so accurately with his 'beaver dam' analogy. Although he neglected to consider the possibility of using inside-out geometry, he certainly understood well that the other method 'depended upon the maker's experience to provide automatically a viable shape of case in which

[96] Henri-Arnaut de Zwolle. Fifteenth Century Burgundian manuscript. A complete English translation of Arnaut's instructions for constructing a clavisimbalum (harpsichord) and other stringed keyboard instruments is given in: Stewart Pollens. *The Early Pianoforte*. Cambridge Univ. Press, 1995. Chapter 1. The full text of the Arnaut manuscript is available in modern reprint and French translation in: G. le Cerf, Ed. *Les traités d'Henri-Arnaut de Zwolle et de divers anonymes*. Editions Auguste Picard, Paris, 1952.

[97] For example: M. Frascari. Contra divinam proportionem. In: K. Williams, Ed. *Nexus II. Architecture and Mathematics.* Edizioni dell'Erba, 1998, pp.65-74.

[98] Henri-Arnaut de Zwolle, op. cit.

to insert string band and action.' [99] We can now see that this amounted to nothing more than knowing the steps of an appropriate geometric construction which guaranteed correct proportional and dimensional relationships in the case. Therefore, seen in the context of historical design principles, the apparently abstruse 'maker's experience' is actually a relatively simple aspect of the maker's skills, something that could have been taught to an apprentice in a few hours.

Stringed keyboard instruments may be conveniently classified according to the orientation of the strings with respect to the action, as either: (i) *parallel*, i.e. fluegel-shapes in which the strings and keys are essentially along the same line; or (ii) *perpendicular*, i.e. those for which strings and keys are essentially perpendicular, although the angle is often oblique rather than exactly orthogonal.[100] The primary practical focus of the design of most perpendicular instruments seems to have been to ensure that the keyboard fit in the proper location with respect to the stringband, i.e. geometry was predominantly determined by mechanical function in such instruments. The usually quite distorted string scaling of perpendicular instruments indicates that the stringband was probably not the generating factor in their design, i.e. they were generally built following a construction based on outside-in geometry. Parallel instruments may be designed easily using either of the geometric approaches. Inside-out parallel instruments such as Italian harpsichords have been described above; outside-in parallel instruments are more likely to have a significant straight section of bentside, because this simplifies the geometry where the string lengths are essentially arbitrary anyway. If the bridge is also parallel to this straight bentside (e.g. a Stein fortepiano), the beginning of that section often corresponds to the point on the bridge where significant scalar shortening begins.

THE CONSTRUCTIVE GEOMETRY OF VIENNESE AND SOUTH GERMAN PIANOS

To illustrate the concepts discussed above, we present some results of an extensive analytical study of the geometry of extant pianos from the Viennese and South German tradition prior to the advent of industrialised manufacturing. A common working practice seems to have been in ubiquitous use throughout this period, from the early five-octave pianos of Silbermann of the 1740s, to the six and one half octave 1840 pianos by builders such as Conrad Graf and J.B. Streicher. The essential aspects – the generating module defined by the width, a rectangular reference framework, and critical points constructed geometrically to define the bentside geometry – are summarized below, with reference to Figure 8.

The module. The width of the piano under the keywell is the generating module for all subsequent dimensions. For reasons that will become apparent, we label the half-width w, and the full width is then $2w$. To provide adequate space for the action frame and the case sides, the definition of the module must have taken into account the keyboard compass and structural details of the inner framing, in particular the thickness of the inner rims. The required dimension was expressed in terms (often a multiple) of a smaller elementary modular unit, i.e. a builder's 'inch', or *Werkzoll*,[101] which, in general, was specific to a particular shop, and was not necessarily associated with any local unit of measure. The *Werkzoll* pervades the entire piano design,

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[99] Hubbard, op. cit., p.215.

[100] These definitions exclude instruments such as clavicithera in which the lines of keys and strings are not in the same plane.

[101] The *Werkzoll* concept is discussed at length in Part II. In one sense this elementary modular unit might be considered the basic generating module for the piano, since all dimensions are ultimately derived from it, including the width. However, we prefer to define the generating module of the piano as the fundamental dimension which is used to construct the case geometry. Since this generating module is defined in terms of the *Werkzoll*, there is no conflict of terminology. To distinguish these two concepts, we use the term *Werkzoll*, or modular inch, for the elementary modular unit, and when we refer to 'modular measurement' we mean dimensions expressed in terms of this modular inch.

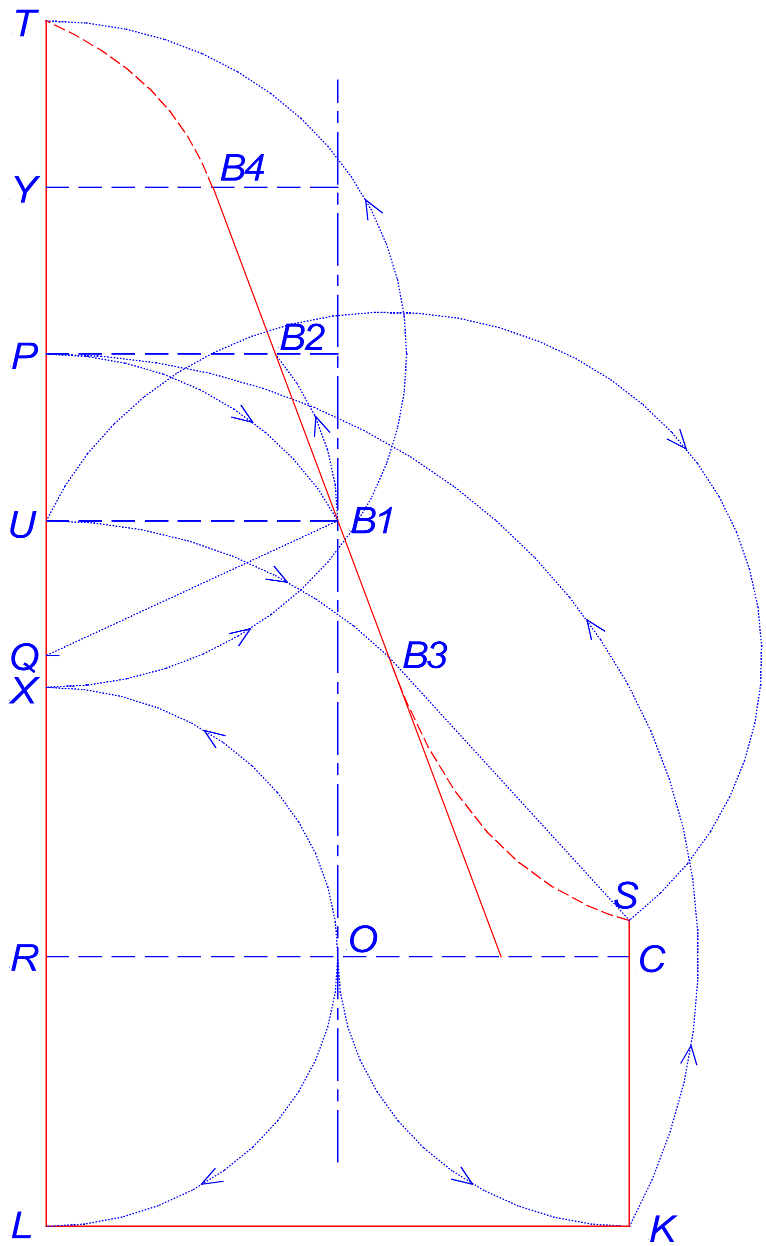

Figure 8. *Geometric construction of points defining the case design of a Stein Phase II fortepiano. Construction sequence is: R, O, C, K, L, X, P, Q, B1, U, B2, B3, T, Y, B4. See text instructions for details.*

including the definition of the generating module. It should be emphasized that the module refers to the width of the bottom boards, which is the outside dimension of the inner case, or framework, in Viennese pianos. The outer case is merely a skin that is glued to the inner framing, overlapping the bottom and contributing nothing to the geometry. This emphasis on the inner case is consistent with a proposed construction sequence based on developing the plan view on the bottom boards for each instrument being made,[102] something that necessarily would have occurred long before any consideration was given to the outside case.

Principal reference lines and keywell. A bottom panel of adequate size was glued up and a long straightedge planed to define the spine line. A reference position R was marked on the spine close to the desired location of the bellyrail, and a perpendicular reference line constructed there. The cheek line SK was constructed parallel to, and distance 2*w* from, the spine line. The location O of the centre line was marked on the perpendicular reference line, i.e. distance *w* from the spine. Finally, a keywell rectangle RCKL with sides 2*w* by *w* was defined, with one side on the perpendicular reference line, by constructing points K and L on the cheek and spine line, in front[103] of the reference line. All of these required points and lines can easily be constructed beginning at point R using dividers set at the distance of the half-width (half-module) *w*. In some five-octave designs the keywell rectangle defined the front edge of the actual keywell, however, in most cases, it was necessary for the actual keywell to be deeper than the rectangle, to provide adequate space for keys of the desired length. Regardless of these considerations the keywell rectangle, and its diagonals and half-diagonals, invariably were used to generate the geometry of the case design.[104]

Bentside geometry. The line which defines the straight part of the bentside was located by constructing two points, B1 and B2, in relation to the keywell rectangle. Two further points were marked on the bentside line, to define the ends of the main straight section, i.e. the points B3, where the treble curve begins, and B4, where the tail section of the bentside begins. The two ends of the bentside are defined by constructing the intersection of the tail section of the bentside with the spine line, T, and of the treble curve of the bentside with the cheek line, S. Thus the orientation and position of the bentside with respect to the keywell rectangle, and its essential geometry, are defined by the constructed points T, B4, B2, B1, B3 and S.

The overall case shape of the piano is precisely defined by these constructed points which determine the fundamental spine, cheek, bentside, and tail geometry.[105] The position of the bellyrail was also located in the geometric framework, by constructing the point where it intersects the spine line, often the same as the reference point R, and also where it intersects the cheek line, if the bellyrail was skewed. Similarly, important bracing members may have been located geometrically (e.g. Stein Phase III, and Schiedmayer constructions). Details of the constructive geometry vary with the builder, the time period, and with the compass of the piano, however the principle of using the keywell rectangle as a reference and generator for other dimensions seems to have been almost universal. The specific geometry determines the relationships between the component parts of the design, and the manner in which the stringband

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[102] When Viennese pianos began to be made with open bottoms in the 1830s, it would have been logical to continue the design practice by constructing the plan on the bench, or on a specially-made bottom which could have been used repeatedly as a template for gluing up the inner frameworks of pianos. This situation is the beginning of the transition to industrialized production.

[103] The terms "front" and "back" refer to orientations with respect to the player's position.

[104] Identification of the location of the main perpendicular reference line in an extant piano is a critical step in analysing the geometry used by the builder.

[105] The short tail section is defined by its endpoints T and B4 (Figure 8). The particular shape may have been curved (e.g. J.A. Stein), angled (e.g. Anton Walter), or perpendicular to the spine (e.g. Nannette Streicher). A straight section is easily constructed simply by joining the endpoints; a curved tail section, and also the treble curve which joins points B3 and S, were probably constructed by using a thin wooden strip as a spline. The specifics of these tail and treble sections are not discussed here, as they are not significant to the holistic geometric design, which is well-defined by the constructed endpoints of these sections.

Instructions for Apprentices

Keywell rectangle. Locate the main reference point R where the bellyrail is to be placed at the spine. Scribe a perpendicular to the spine at R. With the dividers set at the half-module dimension *w* construct the centre point O, and the cheek point C on the perpendicular reference line. Scribe the cheek line through C and parallel to the spine. Construct points on the cheek line K and spine L at distance *w* in front of the perpendicular reference line.

Bentside geometry. *Preliminaries.* With the trammel and an arc centred at R, transfer the diagonal dimension RK to define point P on the spine behind the reference line. Repeat with the same trammel setting, using an arc centred at L, to construct point X on the spine behind the reference line. *Point B1.* Bisect RP to construct the midpoint Q. Construct point B1 where the centre line intersects an arc of radius PQ centred at Q. *Point B2.* Construct a perpendicular to the spine at P. Drop a perpendicular from B1 to the spine at point U. With the trammel set between points U and B1 construct the point B2 where an arc centred at U intersects the perpendicular at P. Scribe and extend the bentside line through points B1 and B2. *Point B3.* With the trammel set at distance UR, construct point B3 where an arc centred at R intersects the bentside line. *Tail point T.* Construct point T where an arc centred at P intersects the spine. The trammel is set either at distance PX (Phase II) or between points P and B1 (Phase III). *Point B4.* Bisect TP to construct the midpoint Y. Construct point B4 where a perpendicular at Y intersects the bentside line.

Bellyrail and front edge of keywell. The reference line defines the front edge of the bellyrail (Phase II), or the back edge (Phase III). Neither piano has a skewed bellyrail. The front of the keywell is defined by line KL (Phase II) or, by transferring the distance RQ to the spine in front of the reference line with the trammel centred at R (Phase III).

fits inside the case with adequate clearance between the bridge and the case edges. The proportional relationships between longitudinal and transverse dimensions ensure that the sounding length of the desired lowest pythagorean note, in many designs corresponding to the position where the linear section of the bentside begins, fits the case. The rate of scalar foreshortening is essentially defined by the angle of the linear section of the bentside.

From the apprentice's point of view, laying out a piano could have proceeded without measuring, or even knowing, any dimensions, other than the generating module (i.e. width of the bottom), and without any understanding of the geometrical basis for the procedure. Simple manipulations of the trammel, using the sides and diagonals of a rectangle and square based on the *w* and 2*w* dimensions, could have been learned and remembered easily with a few hours of training. This only requires knowing a sequence of steps to be followed. To illustrate this point, based on an analysis of extant five-octave pianos by J.A. Stein and his followers from the period 1780 to 1800, we have devised a set of instructions for laying out the characteristic case geometry (see box above). Stein's pianos from this period can be classified into two groups – Phase II and Phase III[106] – with the transition occurring in 1783. Some minor differences between the steps for the two types of Stein piano show how the details of the constructions could vary even for the pianos of a single builder. The geometry for the Phase II design has been derived from analysis of a 1783 Stein in private ownership in Germany, and a plan view line drawing of the 1783 Stein in

[106] Terminology of Michael Latcham, The Pianos of J.A. Stein, Haags Gemeentesmuseum, 1993.

the Boston Museum of Fine Arts.[107] Analysis of the 1783 Stein in the Württembergisches Landesmuseum, Stuttgart,[108] was the basis for the geometric constructions we propose for the Phase III pianos. Furthermore, analyses of various other extant five-octave pianos of Viennese and South German builders who followed Stein supports the proposed constructions. The points referred to in the instructions below are those labelled in the diagrams shown in Figures 8 and 9. As a starting point we assume that the bottom has been prepared and spine planed straight, as described above.

The "Instructions for Apprentices" above are actually very much more easily communicated verbally, than they are as a written description. With minimal training any practically-minded person can follow them and construct the case geometry of a Stein piano with great precision and consistency in just a few minutes. *This demonstrably supports the ease with which an historical craftsman could have designed and constructed stringed keyboard instruments with no need for a reference drawing*.

ANALYSIS OF STEIN GEOMETRY

The bentside geometry of J.A. Stein's pianos, constructed from diagonals and half-diagonals of a 1:2 rectangle, immediately suggests the underlying relationships are based on golden rectangles, i.e. those in which the sides are in proportion of the golden ratio. Two simple constructions are central to the scheme used by Stein:

(1) A pair of interlocked golden rectangles is constructed inside a rectangle with sides in the ratio $\sqrt{5}$ to 1, as shown in Figure 6. The $\sqrt{5}$ side is bisected and a semicircle of radius $\sqrt{5}/2$, centred at the midpoint, is drawn, intersecting the opposite $\sqrt{5}$ side at the points A and F. These points are the corners of the square AFGD that defines the interlocked golden rectangles. The process can be reversed by attaching a square to the long side of any golden rectangle FBCG, resulting in a larger rectangle with sides in ratio $1 + \phi$ to 1, i.e. a larger *golden* rectangle ABCD since this ratio is the same as 1 to ϕ.[109]

(2) A right angle triangle with sides in the ratios $1 : \sqrt{\phi} : \phi$ (approximately 1 : 0.786 : 0.618) can be constructed from a golden rectangle as shown in Figure 5, by transferring the long side of the rectangle to the opposite side, to represent the hypotenuse. The ratio $\sqrt{\phi}$ is historically important.

The Stein geometry is clearly based on these two constructions, as illustrated by the geometric analysis shown in Figures 10 and 11. The two points B1 and B2, which uniquely define both the angle and the position of the straight part of the bentside with respect to the main reference line and spine, are constructed on a $\sqrt{5}$ framework of interlocked golden rectangles scaled in size according to the w dimension: (i) *Position.* The bentside point B1 lies on the centre line at distance $(1+\phi)\ w$ from the main reference line; and (ii) *Angle.* The point B2, which defines the angle of the linear section of the bentside, lies on the end of the $\sqrt{5}$ rectangle, at a distance $\sqrt{\phi}\ w$ from the spine. Therefore, the angle of the line through B1 and B2 has tangent $(1 - \sqrt{\phi}) / \phi$, or about 0.3461, corresponding to an angle of 19.09 degrees.

When analysing an extant piano, the location of B1 can often be identified explicitly, if it lies at the intersection of the centre line with the bentside (ignoring the outer case), as in the

[107] John Koster, *Keyboard Musical Instruments in the Boston Museum of Fine Arts*, Museum of Fine Arts, Boston, 1997

[108] Latcham Op. Cit. dates the Stuttgart Stein as 1788 without any explanation, yet it is clearly dated 1783 in pencil inside. The internal design particulars indicate this piano is an example of Stein's Phase III output, using Latcham's terminology.

[109] The relationship relies on the unique property of the number ϕ that $1/\phi = 1+ \phi$.

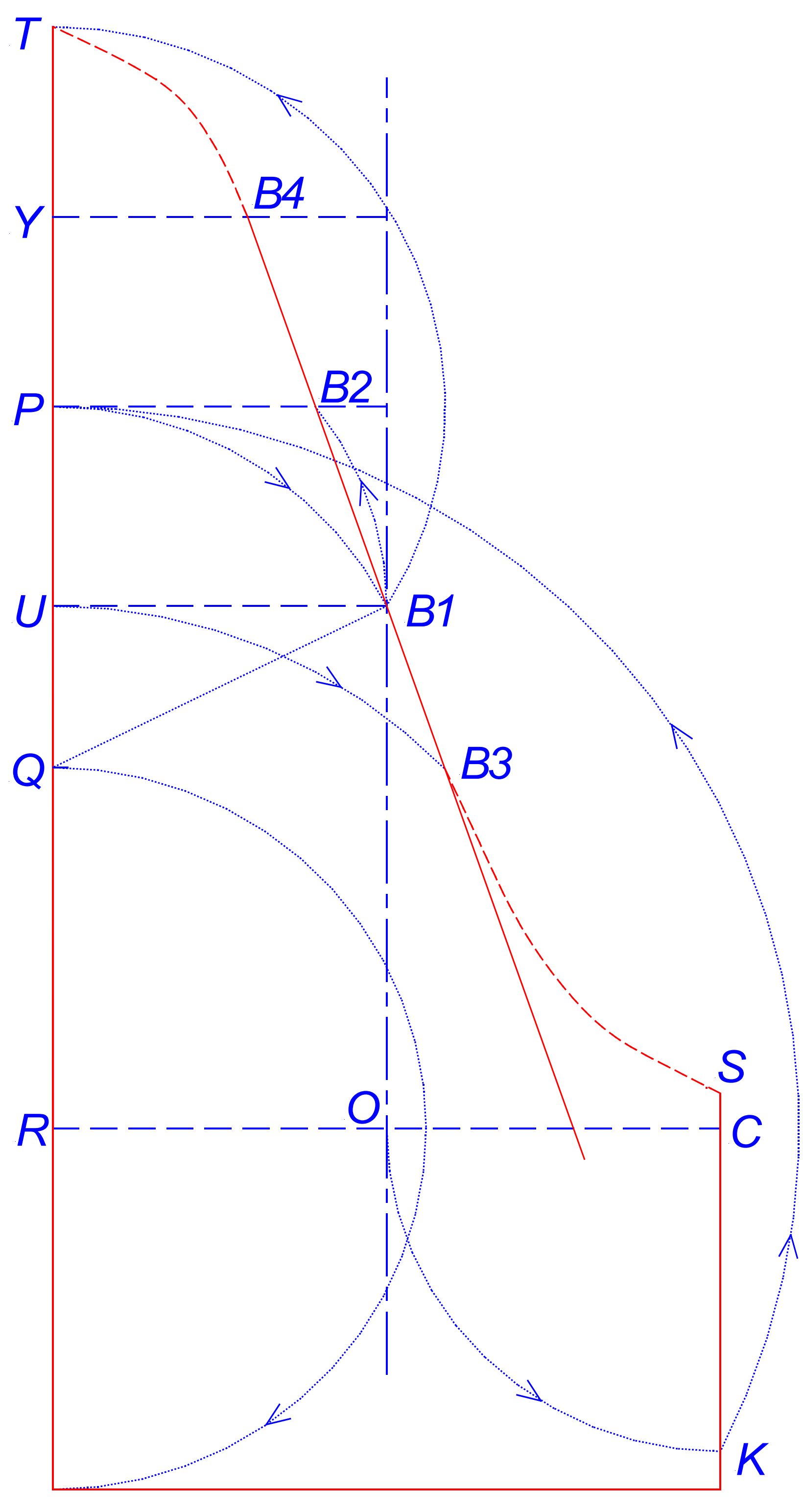

Figure 9. *Geometric construction of points defining the case design of a Stein Phase III fortepiano. Construction sequence is: R, O, C, K, P, Q, B1, U, B2, B3, T, Y, B4. See text instructions for details.*

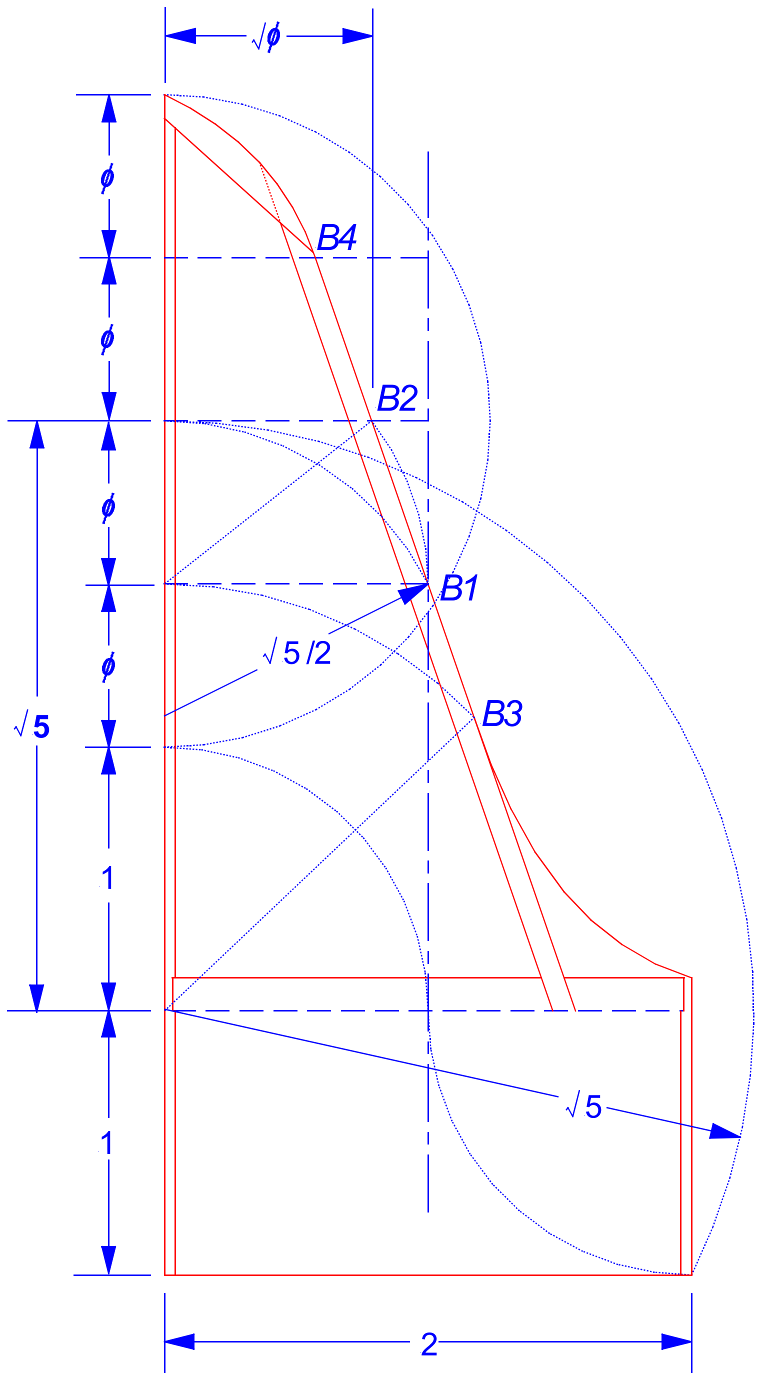

Figure 10. *Analysis of geometry of Stein Phase II fortepiano design. Neither the construction lines, shown dashed, nor the trammel arcs, shown dotted, need have been physically scribed on the bottom, as only the endpoints are important. Bellyrail is behind the perpendicular reference line. Keywell front edge is aligned with keywell construction rectangle. Geometric framework is based on construction of the interlocked golden rectangles in a √5 rectangle (Figure 6) and √ϕ construction (Figure 5).*

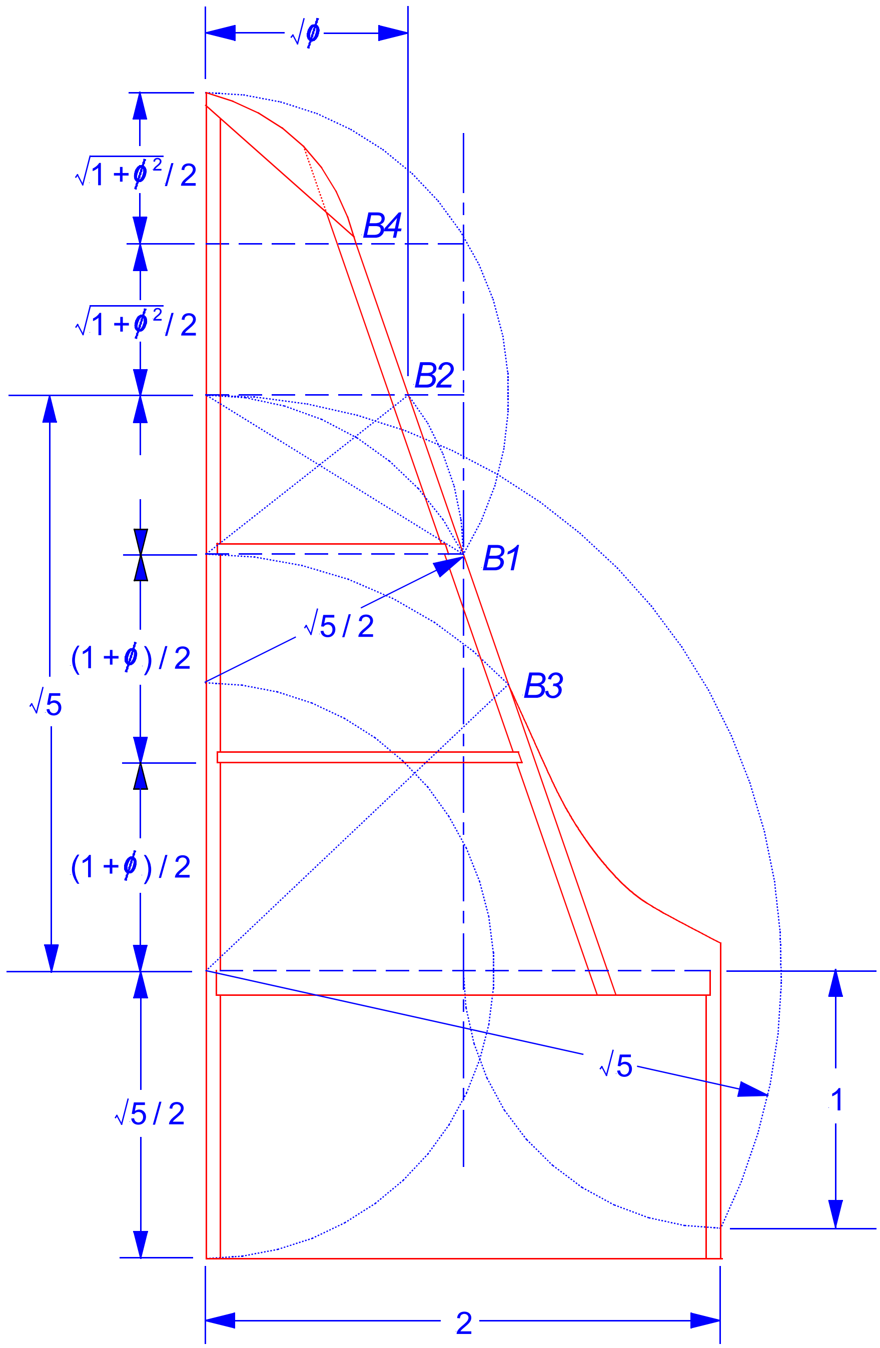

Figure 11. *Analysis of geometry of Stein Phase III fortepiano design. Neither the construction lines, shown dashed, nor the trammel arcs, shown dotted, need have been physically scribed on the bottom, as only the endpoints are important. Bellyrail is in front of the perpendicular reference line. Geometric framework is based on construction of the interlocked golden rectangles in a √5 rectangle (Figure 6) and √ϕ construction (Figure 5). Tail geometry and location of keywell front edge is adjusted as compared to the Phase II design.*

Stein constructions. Measuring a distance $(1+\phi)\, w$ to the front of the piano from the proposed B1 point locates a proposed position for the centre of the perpendicular reference line. This is invariably seen to be consistent with the observed location of the bellyrail and the subsequent independent analysis of the remaining geometry.[110] A piano laid out following the Stein instructions given above will automatically result in a characteristic bentside angle of 19 degrees, consistent with the observed bentside angle we have measured on about 20 extant five-octave Viennese and South German pianos by builders who followed Stein's geometry (e.g. S. Langerer, J.D. Shiedmayer, L. Duelcken, Geschwister Stein). The observed proportional relationships between longitudinal and transverse dimensions of these pianos are consistent with the theoretical ones anticipated, regardless of the absolute size of the piano, which varies according to the size of the module used by a particular builder.[111] For example, the proportional relationships in an extant five-octave piano by J.D. Schiedmayer,[112] which has a half-module *w* of 472mm, are consistent with the geometry of the Stein Phase III construction, except that he has positioned the front of the bellyrail on the reference line, as in the Stein Phase II pianos. In this piano the front crossbrace is positioned with its front edge precisely aligned with the back of the $(1+\phi)\, w$ rectangle. This placing of braces on imaginary lines which have no meaning other than that implied by the geometric constructions we have proposed, is also observed in the Stein Phase III design, for which the front edge of the second crossbrace is aligned with the back of the $(1+\phi)\, w$ rectangle, and the front edge of the front crossbrace is at the position midway between that line and the main reference line (see Figure 11).

The theoretical length of the bottom boards of the two Stein designs can be calculated from the constructions above (refer to Figures 10 and 11):

Phase II. $L = w + w + \phi\,(2w) + \phi\,(2w) = 2w\,(1 + 2\,\phi) = 2\sqrt{5}\, w \sim 4.472\, w$

Phase III. $L = \sqrt{5}\, w/2 + \sqrt{5}\, w + w\,\sqrt{(1+\phi^2)} = (3\sqrt{5}/2 + \sqrt{(1+\phi^2)})\, w \sim 4.530\, w$

The predicted bottom length of 2088 mm, calculated for the extant Phase II piano with an observed *w* of 467, can be compared to the observed length of 2087 mm; for the Stuttgart Phase III Stein, with observed *w* of 465 mm, the predicted length of 2107 mm is identical to the observed measured length of the bottom boards. Comparing Stein's Phase II and III designs with a common module, the ratio of the theoretical lengths is about 1.013, i.e. the Phase III is about 1.3% longer than the Phase II. This amounts to some 25 mm difference on a 2000 mm length. Latcham[113] has reported that the Phase III Stein pianos he has measured are consistently longer than the Phase II pianos, however the actual lengths he reports cannot be used with confidence, since the meaning of 'lengths excluding mouldings' is not entirely clear. For instance, the reported length for the Stuttgart Stein is 2125 mm, while we have consistently recorded a highly accurately determined length of 2136 mm, including the 30 mm cross member glued to the front of the bottom boards. Nevertheless, the geometric theory certainly implies the slightly longer Phase III pianos as observed by Latcham.

To demonstrate the agreement between theory and the observed geometry of extant instruments, Figure 12 shows a photograph of the underside of a soundboard, when it was removed from a 1788 Stein Phase III piano in the Germanisches National Museum, Nürnberg. The outer edge of the inner case follows the outside edge of the soundboard. Superimposed on this photograph is an image of the proposed Stein Phase III geometry, which we derived from measurements of the 1783 Stuttgart Phase III Stein. The proportions of these images have not

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110 Problems associated with interpreting data collected from extant instruments are discussed in Part II.

111 The modules used by different builders are discussed in Part II. These varied as a consequence of differences in both a builder's *Werkzoll* and the convention used to define the width module of a particular piano design in terms of this.

112 Germanisches National Museum, Nürnberg. MIR 1102. Dated 1794.

113 Latcham, op. cit., p.18.

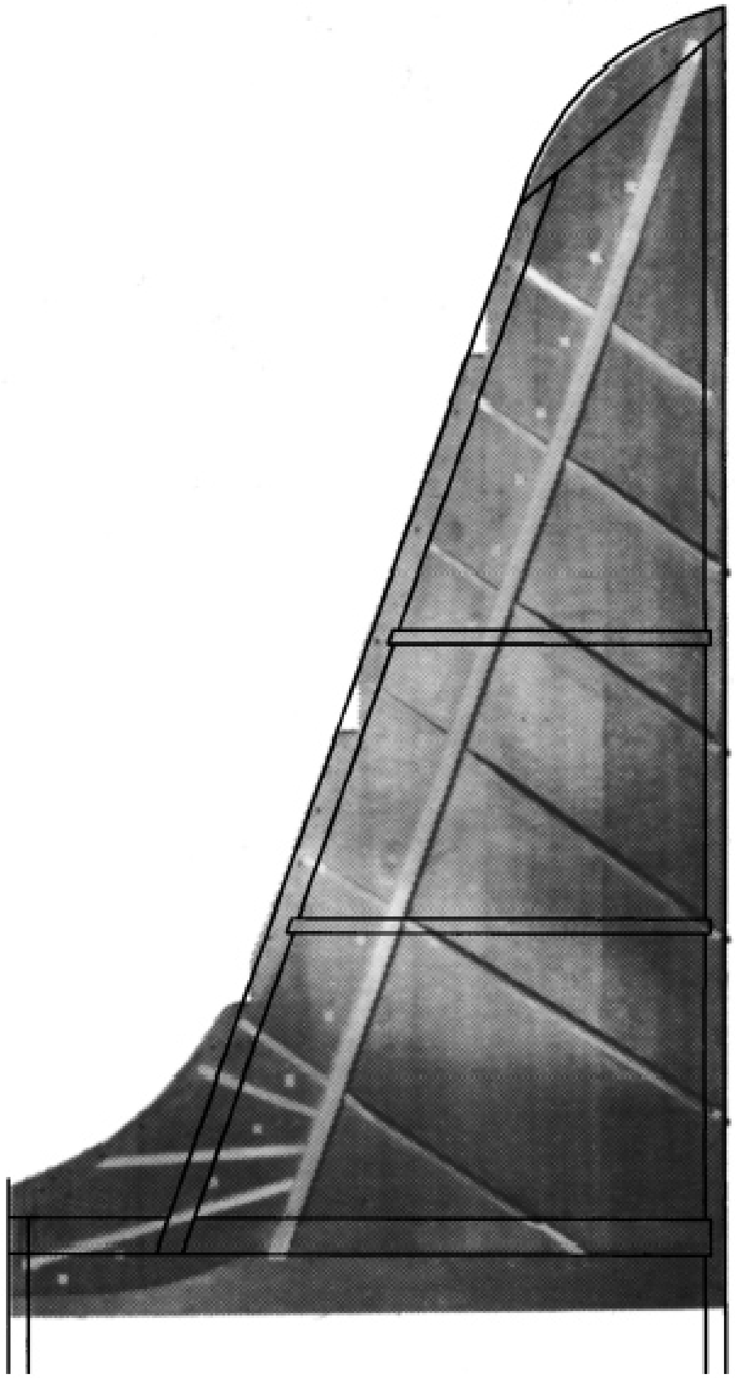

Figure 12. Geometric construction of Stein Phase III fortepiano design (shown in Figure 11) derived from 1783 Stein (Württembergisches Landesmuseum, Stuttgart), superimposed on photograph of the underside of the soundboard removed from a 1788 Phase III Stein (Germanisches National Museum, Nürnberg). Aspect ratios were carefully maintained to avoid any change to proportional relationships.

DISCUSSION AND SUMMARY

In order to gain a genuine understanding of the design and construction methods used by early makers of stringed keyboard instruments, it is essential to establish the general context in which they worked. There is scant historical source material which describes specifically the working methods of instrument makers because the fundamental approach was common to all crafts, and therefore widely-known, and because it was an oral tradition which could not easily be formally recorded. Nevertheless, there are enough general sources, particularly related to architecture and furniture-making, to establish unequivocally the fundamentals of the historical approach to design, from which it is not unreasonable to extrapolate the specifics that pertain to keyboard instrument making, guided by evidence from extant instruments. *For these reasons we have concentrated initially, and in considerable depth, on an analysis of historical design principles and practice in this general context.*

To interpret an historical methodology correctly requires an historical perspective. For example, modern reasoning has led analysts to suppose that contemporary scholarly publications ought to provide a basis for establishing the methods of the craftsman, but, in reality, there was very little connection between their oral traditions and published mathematics. The craftsman's pragmatic techniques were based only on simple, prescriptive procedures (akin to recipes), and were judged solely by the results they produced. Since no concern was given to formal 'correctness', theoretical justification was of no interest to the craftsman. This is a pitfall for the modern analyst, for whom a theoretical basis may seem to be a natural assumption. Historical mathematical sources, including those devoted to so-called practical mathematics, can therefore be misleading, not only because the methods described were not necessarily actually used by craftsmen, but, conversely, many of the methods of the craftsman are likely not included in the mathematical compendia.

It is very widely documented that architecture formed the conceptual and practical foundation for all the constructive crafts. To the historical craftsman, architectural design was synonymous with geometry, therefore we can conclude – and this is also explicitly declared in many historical sources – that geometry was indispensible and central to the practice of any art. Thus geometrical methods provided the common design methodology and links between all the various historical crafts. Geometric procedures were based on simple physical constructions, or approximations of these, which avoided the necessity for almost all algebraic calculation and manipulation of numerical quantities. These constructions, which need not even have been theoretically correct, formed a generally un-recorded oral tradition. Therefore, to discover confidently those which pertain to a particular craft is a difficult task. Historical sources confirm that technical and practical simplicity, and the known physical requirements of the product, were the motivation for defining the geometric procedures; consequently, simplicity and technical requirements should also be the basis for analysis aimed at re-constructing this geometry, rather than a reliance on *ad hoc* assumptions and complicated theories that have their origins in either historical mathematical treatises or modern design methodology.

Two general principles are useful, and can be established from historical sources:

1. The layout tools universally used by the historical craftsman are well-known – the square, compasses (including the large beam compass or trammel), and straightedge – and define the practical limitations under which they worked. These tools strongly imply constructive geometry as the sole basis of working practice for layout and design.
2. A constant theme in virtually all historical design sources is the correct application of proportion, and a harmonious balance between component parts and the whole (reductionism *vs.* holism). In practice this is accomplished by modular design in which proportional geometric relationships between components are defined, and a single dimension, called the module, fixes the absolute size of the whole.

It is important to realize that modular design was primarily a practical means to establish and consistently reproduce both the correct proportions in, and the overall size of, an artifact. Two common misconceptions with regard to this can lead to erroneous conclusions when analysing extant artifacts: (i) The module was chosen purely for practical utility, and there is no reason to suppose *a priori* that it was necessarily defined as a multiple of any local unit of measure, although it certainly may have been if the craftsman found that to be a convenient choice. This problem is explored in detail in Part II; (ii) Proportional relationships are not explicit attributes of historical design, rather they are the implicit consequence of the geometric constructions that were used to define the inter-relationships between the parts of an artifact. The observation and reporting of random proportions, for instance from extant musical instruments, cannot possibly lead to meaningful conclusions, since the critical link to the shop techniques, i.e. the specific steps of the builder's construction, is absent. Proportions must always be causally related to workshop constructions, not viewed as sterile attributes.

Two different practical geometric techniques were used to establish proportions; an artifact may have been laid out with either of these approaches, or possibly a combination of them. Before focusing on the particulars of stringed keyboard instruments, these can be stated first in the most general terms:

> *Direct constructive geometry.* From a pre-determined modular dimension, initially marked in the artifact, shop constructions based on the use of the layout tools above define the spatial relationships between the parts, and their absolute dimensions. Proportions defined in this way may naturally be either rational or irrational, according to whether the constructions transfer orthogonal or diagonal dimensions respectively. The method is most likely associated with laying out a plan view of the artifact as part of the construction process. Dimensional accuracy will generally be highly self-consistent.
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> *Modular measurement.* Sequences of dimensions, determined as multiples of a (small) elementary modular unit, the builder's *Werkzoll*, are explicitly incorporated in an artifact to define the sizes and spatial arrangement of its parts following a prescribed formula. This method generally produces rational proportions, although the motivation for these choices may have been approximation of some irrational proportions, possibly developed previously using a direct geometric construction as the initial design basis. Although there is no reason why modular measurement cannot be applied in practice while constructing a plan view, it is also feasible to work completely independently of any plan view, defining the positions of component parts abstractly in terms of the physical components being assembled.

Applying these two geometric methods to the design of stringed keyboard instruments, we can conclude that direct geometric construction generally implies a layout based on the bottom (e.g. Viennese pianos), while modular measurement may be used in constructions built on the bottom (e.g. Arnaut, Italian harpsichords), or it may proceed without any reference to the bottom, following an abstract sequence of modular dimensions (e.g. Ruckers harpsichords). This leads to the identification of two fundamental design approaches for stringed keyboard instruments: (i) Inside-out, for which the stringband, constructed based on modular measurements, defines the case geometry (e.g. Arnaut; Italian harpsichords in which the bridge outline defines the bentside); and (ii) Outside-in, for which the case geometry is constructed first, with the assurance that the desired stringband will fit inside. Outside-in designs may be based on modular measurement (e.g. Italian harpsichords in which the case outline defines the bridge position), constructive geometry (e.g. Viennese pianos), or a combination of these (perpendicular instruments like virginals). To illustrate the use of constructive geometry, an example of the outside-in approach was described for the layout of *five-octave* pianos of two designs used by J.A.Stein and his followers. These constructions also illustrate how the golden ratio could explicitly become the pervasive element of entire school of instrument making, even though the builders of these pianos almost certainly were not aware of this. *The historical working practice described above for makers of stringed keyboard instruments has been unequivocally established by considering only relevant historical sources, practicalities of workshop methods, the special technical requirements of stringed* *keyboard instruments, and using simple logic to extrapolate the general methods used by all historical craftmen, with no need for troublesome* a priori *assumptions.*

The second part of this article (Modular measurement) begins by examining issues related to collecting and analysing data from extant instruments. This leads to a discussion of the elusive concept of acceptable tolerance for comparing observed dimensions with those of the nominal geometric scheme that is proposed to have been used by the builder. Acceptable tolerance can be quite strict for constructive geometry, and usually there is little doubt when the such a geometric scheme has been successfully reconstructed from an extant instrument (e.g. the Stein school discussed above). However, since modular measurement is naturally prone to dimensional variation, significant random differences can be expected in a group of instruments supposedly following the same nominal scheme. A reasonably large database of similar instruments is necessary to make meaningful conclusions about design practice when the builder has used modular measurement to define the geometry. The remainder of the second part focuses on independent tests for determining the builder's module from analysis of extant instruments. Without considering all the evidence simultaneously – stringspacing, keyboard dimensions, string lengths, small and large case dimensions – a design can often be shown to be consistent with several different modules, with several corresponding different implied geometric methods. By placing exclusive emphasis of a single method, for instance modular measurement, without considering also constructive geometry, or perhaps both in combination, the analyst may fail to find the method which was actually used by the builder of an extant instrument. What may seem obvious on the basis of some *ad hoc* assumption, especially one with no basis in the context of historical design practice, may in fact not be so obvious when viewed in this way. For instance, the derivation of small and large duims (two different inch measures) supposedly used simultaneously by the Ruckers, is not supportable. In fact, a single duim value can quite adequately explain the geometry of the entire output of the Ruckers and many of their contemporaries, and a value for this Ruckers duim, suggested by considering several independent design factors, is reported in Part II. Several instruments which have been previously analysed in the organological literature are re-analysed, and possible alternative constructions, with alternative modular dimensions, are derived. This illustrates the tenuousness of the connection between local unit of measurement and instrument building, and calls into question the use of local units of measurement to associate instruments and builders according to locale.