The Mathematical Foundations of Bondgraphs VI. — Causality and Regularity for Directed Bondgraphs

S.H.Birkett and P.H. Roe
Department of Systems Design Engineering
University of Waterloo
Waterloo, Ontario, Canada

May 9, 2002

Abstract

This article concludes a series on the mathematical foundations of bondgraphs, an abstraction of the concept of bondgraph junction structure (without TF or GY components) used in physical system modelling. Di-bondgraphs, which are bondgraphs for which a direction, indicated by a half-arrow, has been assigned on each bond, are pictorial representations of a dual pair of integral matroids, called the cycle and co-cycle matroid. Regular di-bondgraphs, which are almost always the type used for system models, are those for which the di-bondgraph and its underlying (non-directed) bondgraph represent the same cycle matroid. The assignment of causality for di-bondgraphs, indicated by adding a causal stroke to one end of each bond, is analysed, including a detailed treatment of causal loops and odd loops, with respect to which non-regular and regular di-bondgraphs behave differently. A causal loop may imply invalid causality for a regular di-bondgraph, but not always, however causal loops always give a *valid* causality for non-regular di-bondgraphs. Assignment of causality is shown to be equivalent to selecting a base and co-base for the cycle matroid of the di-bondgraph. Sub-structures called minors, obtained through bond and junction operations analogous to the contraction and restriction operations on matroids, are defined for di-bondgraphs. This simple geometric procedure can be used to construct an algorithm to determine whether a given di-bondgraph is non-regular by manipulating the diagram itself, without any algebraic analysis. A summary of the entire series and its objectives and relevance to the use of bondgraph in physical system modelling is given in conclusion.

Keywords: bond graph, combinatorial, causality, regular, junction structure, theory, linear graph

1 Introduction

This article concludes the presentation of a mathematical theory underlying the use of bondgraph techniques for physical system models. The first four parts [1, 2, 3, 4] established the mathematical foundations for the combinatorial properties of (non-directed) bondgraphs, diagrams consisting of junctions and bonds only and not involving any physical components such as TF or GY. This is an abstraction of the concept of 'bondgraph junction structure' used as the combinatorial basis for constructing a physical system model [5]. The fifth part of the series [6] defined and studied the combinatorial properties of a di-bondgraph, $\bar{B}$, for which each bond of a bondgraph, B, called the underlying bondgraph, has been given a direction (indicated on the diagram by a half-arrow).

A (non-directed) bondgraph is a pictorial representation of a binary matroid, $M(B)$, called the cycle matroid of B, indicated in terms of a binary chain group which generates $M(B)$. The cycle matroid of the dual bondgraph B^*, formed by reversing the labels of the junctions, is called the co-cycle matroid of B. A di-bondgraph represents an integral matroid, $M(\bar{B})$, called its cycle matroid, by indicating an integral chain group which generates it. In general, there is no *a priori* connection between a di-bondgraph and its underlying bondgraph, and these can represent different matroids. However, a special class of (di-)bondgraphs, the regular (di-)bondgraphs, can be defined. These have regular cycle chain groups and are characterized by the property that $M(B) = M(\bar{B})$, i.e. they generate binary and integral representations, respectively, of the same cycle matroid. Regular di-bondgraphs, which are almost always those used for physical system models, are also orientable. They must be either graphic, in the sense that there is an associated graph with the same circuit matroid, or co-graphic, in which case the dual bondgraph has an associated graph, or both graphic and co-graphic. If $\bar{B}$ is graphic the orientation induced by the (power) half-arrows may be used to assign directions to the edges of an associated cycle graph for B, giving an associated cycle di-graph for $\bar{B}$. A non-regular di-bondgraph has a non-regular cycle matroid, for instance the uniform matroid of rank 2 on four elements, represented by $\bar{B}_{2,4}$. The problem of determining the regularity of a di-bondgraph is analysed in this article and a procedure involving manipulation of bonds and junctions is devised to determine the presence of sub-structures related to matroid minors which may determine non-regularity without the need for any algebraic calculation.

Causality for di-bondgraphs is analysed using pseudo-base colouring for bondgraphs [4], which is equivalent to the conventional bondgraph modelling procedure called 'assigning causality' by adding a causal stroke on one end of each bond. Causal loops have been discussed in the bondgraph literature [7, 5, 8], however, as shown here, some of these references contain incorrect statements since they fail to recognize that non-regular and regular bondgraphs behave differently with respect to causal loops.

The mathematical background needed for the analysis is summarized in appendices to the previous articles of the series: matroid theory [3], base and co-base results for matroids [4], and chain groups and their matroids [6]. For convenience, the important definition of a matroid minor is repeated here in the Appendix, together with a theorem used in Section 3. The definitions, notation, and results of the first five parts of this series [1, 2, 3, 4, 6] are used freely. We will continue to denote junctions with the alphabetic symbols 'p' and 's', for the same reasons as discussed in the previous articles [6], i.e. to avoid confusion which can follow from using the conventional bondgraph model numerical '0' and '1' junction labels in our theoretical context. The convenient juxtaposition notation for sets of bonds will also be used: for example '1256' denotes the set with elements 1, 2, 5 and 6. This notation simplifies that for a set whose elements are sets themselves. So, for instance, a collection of circuits written conveniently as $\{12, 134, 45\}$ using juxtaposition would be written $\{\{12\}, \{134\}, \{45\}\}$ using conventional notation. The symbol $\sharp S$ denotes the number of elements of the set S, called the *cardinality* of S.

2 Bases and Causality for Di-Bondgraphs

The vectors with integral coefficients used to represent the chains of an integral chain group do not form a vector space, since the scalars are not from a field but rather the ring of integers, denoted by Z. A set of vectors satisfying the axioms of a vector space with scalars from a ring R is called an *R-module* and, because of the special properties of the ring Z, Z-modules have similar properties to those of a vector space. In particular all the vector space results of [1, 2] are also true for a Z-module, however the terminology used is somewhat different. For example, the terms corresponding to dimension and basis of vector space are called *rank* and *free base* for a Z-module; A *submodule* of a module corresponds to a subspace of a vector space.

An integral chain group on a set S is associated with a Z-module, the collection of vectors with integer coefficients corresponding to the elements of S, used to represent the chains. If N is an integral chain group the associated Z-module is denoted by N also and the 'vectors' are referred to as 'chains'; the meaning should be clear from the context.

Let $\bar{B}$ be a di-bondgraph with bond set E, the set of all the n bonds of $\bar{B}$, and external bond set E_e, the set of all the e external bonds of $\bar{B}$. The *bond chain group* $N(\bar{B})$ is the chain group on E generated by the chains f_k, $k = 1, \cdots, n$, where f_k has support k. The *external chain group* $N_e(\bar{B})$ is the restriction of N to the set E_e. When regarded as a Z-module N has rank n and N_e has rank e.

Theorem 1 Let $\bar{B}$ be a simple proper di-bondgraph. The rank of N_s is

$$\rho(N_s) = s - p + n - e_s$$

and the rank of the quotient module N/N_s is

$$\rho(N/N_s) = p - s + e_s.$$

Dually the rank of N_p is

$$\rho(N_p) = p - s + n - e_p$$

and the rank of the quotient module N/N_p is

$$\rho(N/N_p) = s - p + e_p.$$

Proof: The proof of this result is the same as that of *Theorem IV* of [4].

A *pseudo-base colouring* of a di-bondgraph $\bar{B}$ is a pseudo-base colouring of the underlying bondgraph B. The set β is a *pseudo-base set* of $\bar{B}$ if it is a pseudo-base set of B. Corresponding to each element $\beta_k \in \beta$ define a chain $\hat{\beta_k}$ with support β_k. The chains $\hat{\beta_k}$ generate an integral chain group denoted by N_β.

Theorem 2 Let $\bar{B}$ be a simple proper di-bondgraph with pseudo-base set $\beta = \beta_1\beta_2\cdots\beta_r$, where each β_k is a single (external) bond of $\bar{B}$. The *Z*-module N_β is a submodule of N_e of rank $r = p - s + e_s$. The set of chains $\hat{\beta_k}$ is a free base for N_β.

Proof: The result follows by analogy with *Theorems VII* and *VIII* of [4].

A circuit of the underlying graph of a di-bondgraph $\bar{B}$ is called a *loop* of $\bar{B}$. If no two adjacent junctions of a loop are of the same type it is called a *proper loop*. A k-loop is a proper loop which consists of k internal bonds and, since no two junctions of the same type are adjacent, k must be even. If an even number of bonds are directed clockwise (or equivalently counterclockwise) in a loop, it is called an *even loop* of $\bar{B}$. If an odd number of bonds are directed clockwise (and hence counterclockwise) in a loop, it is called an *odd loop* of $\bar{B}$. A *minimal odd loop* is an odd 4-loop in which each junction has at least one incident external bond.

Lemma 1 Let $\bar{B}$ be di-bondgraph which is a proper loop. The proper contraction of $\bar{B}$ is an odd loop if and only if $\bar{B}$ is an odd loop.

Proof: There are no two adjacent junctions of the same type, since $\bar{B}$ is a proper loop, so all contractions of internal bonds occur in pairs. Contracting an internal p-junction with two oppositely directed internal bonds leaves two adjacent s-junctions and the internal bond joining these may then be contracted. The net effect is to remove two bonds similarly directed with respect to the loop, which does not change the even or odd character of the loop. Contracting a pair of internal junctions of the type shown in Figure 1 does not change the even or odd character of the loop.

Let $\bar{B}$ be a di-bondgraph with a given pseudo-base colouring. A *causal loop* of $\bar{B}$ is a loop of alternately red and white internal bonds of $\bar{B}$ (called a red-white loop in [4]).

Theorem 3 Let $\bar{B}$ be a simple proper di-bondgraph. If β is a pseudo-base set for $\bar{B}$, defined by a pseudo-base colouring which has no even causal loop, then

$$N_s \cap N_\beta = \{0\}.$$

Proof: Suppose there is a non-zero chain $\alpha \in N_s \cap N_\beta$. Then $\mathrm{supp}(\alpha) \subseteq \beta$ and α is a linear combination of the junction chains of $\bar{B}$. The argument used in the proof of *Theorem XI* of [4] shows that there must be a causal loop in $\bar{B}$ as indicated in Figure 2. It is to be proved that the causal loop must be an even causal loop.

In the linear combination of junction chains for α it is possible to choose all the scalars to be positive (*e.g.* $-b_1\bar{b_2} = +\bar{b_1}b_2$) and assume that this has been done. There are non-zero coefficients corresponding to the internal bond o_m, in two of the chains of this sum, those for the two junctions m and $m+1$, on which o_m is incident. It is necessary that no internal bond is in $\mathrm{supp}(\alpha)$, so, in the chains for these adjacent junctions, the coefficients must be opposite in sign and ± 1.

Define $\sigma_1 = 1$, and σ_m, $m = 2, \cdots, 2j$, as follows: if, in the sum for α, the coefficient of o_m in the chain on junction m is the same sign as the coefficient of o_{m-1} in the chain on junction $m-1$ then σ_m is defined to be 1; if the signs are opposite then σ_m is defined to be -1. It is the values of σ_m which determine whether the signs of the coefficients for bond o_{2j} are opposite in the chains on junction $2j$ and junction 1, which is required so that no internal bond is in $\mathrm{supp}(\alpha)$. In particular this will occur if and only if $\prod_{m=1}^{2j} \sigma_m = 1$.

The values of σ_m are now related to the directions on the bonds of the causal loop. For a p-junction $\sigma_m = 1$ since the coefficients of the bonds in a p-chain are always opposite in sign. For an s-junction $\sigma_m = -1$ if and only if the bonds o_{m-1} and o_m are oppositely directed, since then the appropriate coefficients are opposite in sign. Thus $\prod_{m=1}^{2j} \sigma_m = 1$ if and only if there are an even number of s-junctions with both of the causal loop incident bonds directed into the junction or both directed out of the junction. Considering the directions of the bonds with respect to a reference direction for traversing the loop, the required condition is an even number of direction changes with respect to the loop occurring on s-junctions. A loop with every bond directed in the same reference direction is an even loop and, in particular, has no direction changes on s-junctions. Each time the direction of a bond is altered an s-junction direction change is introduced and so there will be an even number of s-junction direction changes if and only if an even number of bond directions are altered, *i.e.* the loop is an even loop. In conclusion, combining the results above, no internal bond can be in $\mathrm{supp}(\alpha)$ if and only if the causal loop is an even causal loop.

Corollary 1 Let $\bar{B}$ be a simple proper di-bondgraph. If β is a pseudo-base set for $\bar{B}$, defined by a pseudo-base colouring which has no even causal loop, then

$$N_{cy} \cap N_\beta = \{0\}.$$

Theorem 4 If $\bar{B}$ be a simple proper di-bondgraph and if β is a pseudo-base set defined by a pseudo-base colouring of $\bar{B}$ which has no even causal loop, then

$$N = N_s \oplus N_\beta.$$

Proof. N_s and N_β are submodules of $N(\bar{B})$. By Theorem 1

$$\rho(N_s) = s - p - e_s + \rho(N)$$

and by Theorem 2

$$\rho(N_\beta) = p - s + e_s.$$

Thus N_s and N_β are submodules of N whose ranks add to $\rho(N)$ and, by Theorem 3, these submodules have zero intersection. The result of the theorem follows.

Corollary 2 Let $\bar{B}$ be a simple proper di-bondgraph. If β is a pseudo-base set defined by a pseudo-base colouring of $\bar{B}$ which has no even causal loop, then

$$N_\beta \cong N/N_s.$$

Theorem 5 Let $\bar{B}$ be a proper simple di-bondgraph. A base set β, defined by a pseudo-base colouring of $\bar{B}$ which has no even causal loop, is a base of $M(\bar{B})$.

Proof: The proof is analogous to the proof of *Theorem XI* of [4].

A pseudo-base set β of a di-bondgraph $\bar{B}$ which is a base of the cycle matroid of $\bar{B}$ is called a *base set* of $\bar{B}$ and the defining pseudo-base colouring is a *base colouring* for $\bar{B}$.

Theorem 6 Let $\bar{B}$ be a simple proper di-bondgraph. If a base colouring of $\bar{B}$ exists then

$$\rho(\bar{B}) = \rho(N_{co}) = p - s + e_s$$
$$\rho^*(\bar{B}) = \rho(N_{cy}) = s - p + e_p.$$

Proof: These results follow from *Theorem XII* and *XIII* of [4].

Theorem 7 The cycle and and co-cycle matroids of a simple di-bondgraph are dual matroids:

$$(M^*(\bar{B}))^* = M(\bar{B}).$$

Proof: If the di-bondgraph is orthogonal then Theorem 5 of [6] may be combined with Theorem 6 to conclude that the cycle and co-cycle chain groups are orthogonal complements. Then, by *Theorem III* of [3], the cycle and co-cycle matroids of $\bar{B}$ are dual matroids. To consider a pseudo-orthogonal di-bondgraph suppose this is obtained from an orthogonal di-bondgraph $\bar{B}$ by reversing the direction of some of the external bonds. Changing the direction of an external bond which is on an s-junction alters the sign of the corresponding entry in every chain of $N(\bar{B})$ but it does not alter the supports of these chains. Similarly changing the direction of an external bond which is on a p-junction alters the sign of the corresponding entry in every chain of $N^*(\bar{B})$ but it does not alter the supports of these chains. These comments follow from the fact that an external bond occurs in only one chain of a minimal generating set. Hence changing the directions of external bonds does not change the cycle and co-cycle matroids, only their chain group representations. Thus the cycle and co-cycle matroids of a pseudo-orthogonal di-bondgraph are dual matroids.

Example 1 The cycle chain group for the di-bondgraph $\bar{B}$, shown in Figure 3, is given below:

	1	2	3	4	5	6	
x_1	1	1	0	0	0	1	126
x_2	0	1	1	1	0	0	234
x_3	0	0	0	1	1	1	456
$x_1 - x_2$	1	0	-1	-1	0	1	1346
$x_1 - x_3$	1	1	0	-1	-1	0	1245
$x_2 - x_3$	0	1	1	0	-1	-1	2356
$x_1 - x_2 + x_3$	1	0	-1	0	1	2	1356
$x_1 + x_2 - x_3$	1	2	1	0	-1	0	1235
$x_1 - x_2 - x_3$	1	0	-1	2	-1	0	1345

The co-cycle chain group for $\bar{B}$ is:

	1	2	3	4	5	6	
y_1	1	-1	1	0	0	0	123
y_2	0	0	1	-1	1	0	345
y_3	1	0	0	0	1	-1	156
$y_1 - y_2$	1	-1	0	1	-1	0	1245
$y_1 - y_3$	0	-1	1	0	-1	1	2356
$y_2 - y_3$	-1	0	1	-1	0	-1	1346
$y_1 - y_2 + y_3$	2	-1	0	1	0	-1	1246
$y_1 + y_2 - y_3$	0	-1	2	-1	0	1	2346
$y_1 - y_2 - y_3$	0	-1	0	1	-2	1	2456

A pseudo-base colouring of $\bar{B}$ is given and, since there is no even causal loop, it is a base-colouring. The corresponding base set $\beta = 135$ is a base of the cycle matroid of $\bar{B}$, the matroid with circuits the supports of the elementary chains of the cycle chain group:

$$\{126, 234, 456, 1346, 1245, 2356, 1356, 1235, 1345\}.$$

Compare $M(\bar{B})$ to the cycle matroid of the underlying bondgraph $M(B)$ with circuits:

$$\{126, 234, 456, 1346, 1245, 2356, 135\}.$$

The base β of $M(\bar{B})$ is a circuit of $M(B)$. The last three co-cycles listed above in the co-cycle chain group of $\bar{B}$ may be used to obtain an integral representation of the cycle matroid $M(\bar{B})$:

1	3	5	2	4	6
2	0	0	-1	1	-1
0	2	0	-1	-1	1
0	0	2	1	-1	-1

Since the representation is over the ring Z and not a field, it is not possible to obtain a standard representation which, over the field of rational numbers would be derived from this integral representation by multiplying each row by $1/2$. Nevertheless the integral representation above explicitly shows that the set 135 is a base of $M(\bar{B})$.

The last three chains listed in the cycle chain group of the di-bondgraph of Example 1 are elementary and there are no primitive chains with the same support. Thus the cycle chain group is non-regular. It would be incorrect to conclude from this, though, that the cycle matroid is also non-regular, because it might be a regular matroid of a non-regular chain group, like that of Example 6 of [6]. The cycle matroid is, in fact, non-regular, for the minor obtained by deleting bond 2 and contracting bond 3 is isomorphic to $U_{2,4}$ since it has the following circuits:

$$\{456, 146, 156, 145\}.$$

Example 2 below shows how to obtain this minor directly by manipulating the bondgraph diagram itself.

Let N be an integral chain group with a matroid of rank r and co-rank q and let $\beta = 1 \cdots r$ be a base of M. If $c_1^*, \cdots, c_r^*$, generating chains for $N^\perp$, are chosen corresponding to the standard integral representation of M with respect to the base β, they are called *β-generating chains* for $N^\perp$. In this case, if $c_j^* = (c_{j1}^*, \cdots, c_{jn}^*)$, then, for $r \geq k$,

$$c_{jk}^* = a\delta_{jk} = \begin{cases} a & j = k \\ 0 & j \neq k \end{cases}$$

where a is an integer and δ_{jk} is the *Kronecker delta.* Dually, q *β-generating chains* for N may be chosen corresponding to the standard representation of M^* so that, if $c_j = (c_{j1}, \cdots, c_{jn})$, $j = 1, \cdots, q$, then $c_{jk} = a\delta_{jk}$, for $r < k$. Example 1 shows how these definitions apply to di-bondgraph matroids and their bases.

3 Regularity of Di-Bondgraphs

Let $\bar{B}$ be a di-bondgraph and b any external bond. The operation of *deleting* b from $\bar{B}$ consists of removing b from $\bar{B}$, if it is on a p-junction, or removing b, the junction and all other bonds on that junction, if b is on an s-junction. The di-bondgraph obtained is denoted by $\bar{B} \times b$. The operation of *contracting* b from $\bar{B}$ consists of removing b from $\bar{B}$, if it is on an s-junction, or removing b, the junction and all other bonds on that junction, if b is on a p-junction. The di-bondgraph obtained from this dual operation is denoted by $\bar{B} \circ b$. A di-bondgraph is called a *minor* of $\bar{B}$ if it is obtained from $\bar{B}$ by any combination of contractions and deletions of external bonds. If B is a bondgraph, the contraction and deletion minors of B are defined and denoted similarly.

Theorem 8 Let b be an external bond of a di-bondgraph $\bar{B}$. Then

$$\begin{aligned} (\bar{B} \times b)^* &= \bar{B}^* \circ b \\ (\bar{B} \circ b)^* &= \bar{B}^* \times b \end{aligned}$$

Proof: This result follows immediately from the definitions.

Theorem 9 Let b be an external bond of a di-bondgraph $\bar{B}$ with cycle matroid M. Then

$$\begin{aligned} M(\bar{B} \times b) &= M \times b \\ M(\bar{B} \circ b) &= M \circ b \end{aligned}$$

In particular the cycle matroid of any minor of $\bar{B}$ is a minor of M.

Proof: Removing an external bond b from a p-junction removes one chain from a minimal generating set of junction chains for $M(\bar{B})$, a chain of type $b\bar{o}$, where o is an internal bond incident on the same p-junction as b. Removing an s-junction with a single incident bond b, together with the incident internal bonds, removes from a minimal generating set of junction chains a single chain involving only b and the internal bonds. Thus the circuits of $M(\bar{B} \times b)$ will consist of those circuits of M which do not contain b, *i.e.* $M(\bar{B} \times b) = M \times b$. The following calculation establishes the other result:

$$\begin{aligned} M(\bar{B} \circ b) &= M^*((\bar{B} \circ b)^*) \\ &= M^*(\bar{B}^* \times b) \\ &= (M(\bar{B}^* \times b))^* \\ &= (M(\bar{B}^*) \times b)^* \\ &= (M^*(\bar{B}) \times b)^* \\ &= (M^*(\bar{B}))^* \circ b \\ &= M(\bar{B}) \circ b \end{aligned}$$

These equalities follow from the definitions of the junction chain group and its dual [6], Theorem 8, the dual result proved above, Theorem M-1 and Theorem 7.

Theorem 10 A bondgraph is non-regular if and only if it has the Fano bondgraph or its dual as a minor.

Proof: This follows from *Theorem M-8* of [6].

Theorem 11 A di-bondgraph is non-regular if and only if it has a minor which is topologically equivalent to $\bar{B}_{2,4}$. If $\bar{B}$ has a minor whose proper contraction is a minimal odd loop then $\bar{B}$ is non-regular.

Proof: A di-bondgraph is non-regular if and only if its cycle matroid has a minor isomorphic to $U_{2,4}$, by *Theorem M-9* of [6]. This is equivalent to the existence of a di-bondgraph minor whose cycle matroid is isomorphic to $U_{2,4}$. Now, suppose $\bar{B}$ has a minor whose proper contraction, which is equivalent, is a minimal odd loop. The minor of this minimal odd loop obtained by removing all but one external bond from each junction is $\bar{B}_{2,4}$ on the four remaining bonds.

Example 2 In the remarks following Example 1 it was noted that the cycle matroid of the di-bondgraph $\bar{B}$ of that example is non-regular since it has a minor isomorphic to $U_{2,4}$. The contraction and deletion operations can be used on $\bar{B}$ to extract the minor $\bar{B}_{2,4}$ from $\bar{B}$, thus demonstrating explicitly that the di-bondgraph is non-regular. $\bar{B} \times 2 \circ 3$, the minor of $\bar{B}$ obtained by removing bonds 2 and 3 from the di-bondgraph, is shown in Figure 4 together with its proper contraction. This minor is clearly the uniform di-bondgraph of rank 2 on the four elements 1456 and thus $\bar{B}$ is non-regular.

Theorem 12 If a di-bondgraph is non-regular it has an odd loop.

Proof: A non-regular di-bondgraph has a minor whose proper contraction is a minimal odd loop, by Theorem 4. This proper contraction must have originated from an odd loop of the di-bondgraph, since it is a loop and is odd if and only if the loop it is contracted from is odd, by Lemma 1.

Example 3 Consider the bondgraph B shown in Figure 5. There are three distinct loops in B and it is not possible to direct the bonds of B without obtaining at least one odd loop. Three possibilities are shown in Figure 5. It is possible to direct B so that every loop is odd and, in particular, there are no even loops. Thus a pseudo-base colouring of B must always be a base colouring, regardless of the presence of causal loops, since a directed version of B has no even causal loops. These results amplify the comments in Remark 7, following Example 7 of [4], which provides an example of a bondgraph for which causal loops gives a valid assignment of causality. Each of the directed versions of B is a regular di-bondgraph in spite of the fact that they all have at least one odd loop.

Example 2 shows that the converse to Theorem 12 is false. A di-bondgraph with an odd loop may be a regular di-bondgraph.

Theorem 13 If B is a non-regular bondgraph then every directed version of B is a non-regular di-bondgraph.

Proof: If B is a non-regular bondgraph it must contain the Fano bondgraph or its dual as a minor. Every junction of the Fano bondgraph or its dual has one external bond and it is not possible to direct the Fano bondgraph or its dual so that every 4-loop is even. Thus every directed version of B contains a minor with an explicit minimal odd 4-loop and so it must be a non-regular di-bondgraph.

Example 4 In Figure 6 a directed version of the dual Fano bondgraph is shown. The cycle chain group of this di-bondgraph $\bar{B}$ is given below:

	1	2	3	4	5	6	7	
x_1	1	0	0	1	0	1	-1	1467
x_2	0	1	0	1	1	0	-1	2457
x_3	0	0	1	0	1	1	-1	3567
$x_1 - x_2$	1	-1	0	0	-1	1	0	1256
$x_1 - x_3$	1	0	-1	1	-1	0	0	1345
$x_2 - x_3$	0	1	-1	1	0	-1	0	2346
$x_1 - x_2 + x_3$	1	-1	1	0	0	2	-1	12367
$x_1 - x_2 - x_3$	1	-1	-1	0	-2	0	1	12357
$x_1 + x_2 - x_3$	1	1	-1	2	0	0	-1	12347

The supports of the elementary chains, shown on the right of the chain group, are the circuits of the cycle matroid of $\bar{B}$.

Examining the di-bondgraph determines that contracting bond 7 leaves an odd 6-loop with one external bond on each junction. A minimal odd loop is obtained as the proper contraction after removing the external bonds on any two adjacent junctions of the odd 6-loop. Choosing bonds 3 and 6 gives the minor $\bar{B} \times 6 \circ 37$ which is $\bar{B}_{2,4}$ on the bonds 1245. Therefore the conclusion is that $\bar{B}$ is a non-regular di-bondgraph. The bondgraph operations used to obtain this minor are shown in Figure 6: the central p-junction and all its bonds are removed, as well as bonds 3 and 6, and the remaining di-bondgraph is simplified to its proper contraction.

4 Discussion and Conclusions

In order to apply bondgraphs as physical system models it is, of course, not necessary to use or understand the bondgraph theory in terms of matroids and chain groups as presented in this series. However, even though it is possible to construct a bondgraph physical system model based on simple basic principles and *ad hoc* rules, a rigorous analysis of the theoretical combinatorial foundation of the methodology cannot but help in encouraging a more widespread acceptance of the techniques. Irrespective of these considerations, the unfortunate independent development of linear graph and bondgraph modelling techniques, and misunderstanding of the relative capabilities and important connections between these two combinatorial methodologies, are serious issues that need to be addressed. In this series we have attempted to demonstrate that, as models of discrete physical systems, there is essentially no intrinsic difference between the combintorial properties of these two types of model, i.e. *linear graphs and bondgraphs have identical combinatorial capabilities.*

The properties of graphic and co-graphic bondgraphs and their relationship with associated graphs have been explored. A complete theoretical mathematical foundation has been presented for the use of bondgraphs as pictorial representations of combinatorial information in physical system models [1, 2, 3, 4, 6, 9]. This analysis also establishes the precise combinatorial requirements for the structure of a spatially discrete physical system model, viz a dual pair of integral matroids, the cycle and co-cycle matroid, which define sets of algebraic (spatial) constraints between system variables and between the dual system variables. To take into account the polarities required for physical measurements, these relations are expressed using integral representations of the cycle and co-cycle matroids. A di-bondgraph gives a direct pictorial representation of this information by showing a set of generating integral chains, whereas a di-graph does so indirectly, by indicating an orientation of the structure matroids.

An orthogonal graphic di-bondgraph, in which every external bond is directed outward, is equivalent to a di-graph and provides the same orientation of the matroid structure. A pseudo-orthogonal graphic and co-graphic di-bondgraph, in which the half-arrows on external bonds may be directed in either direction, is equivalent to a pair of di-graphs, with directions on edges corresponding to inwardly-directed external bonds reversed between the two di-graphs. Strict orthogonality of primal and dual structure is not an inherent aspect of system structure, as shown by the satisfactory use of pseudo-orthogonal di-bondgraphs as physical system models. All that is required is the pair of integral representations of the cycle matroid and its dual.

A regular di-bondgraph, as used in almost every bond graph physical system model, defines the same matroid as its underlying bondgraph. In this case, the di-bondgraph is graphic or co-graphic, or both, and the half-arrows generate an orientation of the cycle and co-cycle matroid structure. Causality for di-bondgraphs has been analysed using pseudo-base colourings, which are equivalent to the conventional causal strokes added to the ends of the bonds of a physical system bondgraph. This procedure can define a base and co-base for the cycle matroid of the di-bondgraph. For regular di-bondgraphs, a causal loop, i.e. a loop in which no adjacent bonds have the causal stroke on the same junction, may imply invalid causality, but this is not always so, contrary to what is generally stated in the bondgraph literature for bondgraph junction structures. For non-regular di-bondgraphs causal loops always give a *valid* assignment of causality.

The problem of determining the regularity of a di-bondgraph was discussed and a close link between bondgraphs and matroids established in terms of the sub-structures called minors. Through the use of bondgraph operations analogous to the restriction and contraction operations on matroids, the regularity or non-regularity of a di-bondgraph can be determined by geometric manipulation of the diagram itself, without the need for any algebraic analysis. Non-regular (non-directed) bondgraphs, those with non-regular cycle matroids such as the Fano matroid or its dual, are not orientable and have no integral representation. Since the latter characteristic is required to represent system variable polarities, non-regular bondgraphs cannot be used in physical system models. Directing the bonds of a non-regular bondgraph must produce a non-regular di-bondgraph, since otherwise it would provide an integral representation of the cycle matroid of the original bondgraph, which does not exist. Therefore, the combinatorial structure represented by a non-regular di-bondgraph can have no relationship to that of the original bondgraph. As for any di-bondgraph, a non-regular di-bondgraph does provide a pair of integral representations and these are occasionally seen in constructing physical system models, for instance for an underspecified system such as a differential gear. Such rare applications are the only ones for which an equivalent linear graph model cannot be constructed using only combinatorial relationships. Such a physical system can, of course, be modelled with a linear graph using physical components such as a transducers, which we have excluded from consideration in developing the pure combinatorial theory.

In physical system modelling, pictorial devices such as bondgraphs or linear graphs, are required only at the input stage where a description of the system structure must be given. For model simplification in the formulation of system equations, combinatorial analysis is best performed directly in terms of the matroid repesentation and this entire computational procedure may be automated. The linear graph or bondgraph diagrams themselves have no relevance to the intrinsic computational aspects of the modelling methodology. It can be concluded that either method may actually be regarded as a special case of a more general abstract combinatorial method, which uses an algebraic structure consisting of a pair of dual matroids and their integral representations, for a complete representation of the system structure and physical variables, for formulation and solution of equations, and for interpretation of the results [10]. Linear graph models have always obscured this fact, due to the emphasis and predominance of graph vertices (or meshes), which are extraneous constructions that have no relevance to the definition of system structure. In this respect bondgraphs are closer conceptually to the abstract methodology. On the other hand, vertices do provide some important computational advantages and powerful formulation techniques that have traditionally not been used in bondgraph formulations. It has been shown that a vertex-like device can be easily contructed for a graphic or co-graphic bondgraphs [9], thereby demonstrating that graphs are computationally no better than bondgraphs in this respect.

An abstract combinatorial methodology that uses only matroid concepts for modelling physical systems has been developed [10]. All of the well-known techniques of linear graph modelling – branch, chord, branch-chord, sparse tableaux, nodal and mesh equations, mixed nodal tableau, *etc.* – have counterparts in this abstract setting. In particular, these may also be applied to any bondgraph model, and bondgraph formulations need not be limited to the conventional state-space approach. These different formulations correspond to various types of coordinatization of the combinatorial structures of the system. A deep understanding of these methods, and, indeed, of the essence of the combinatorial modelling techniques themselves, may be achieved using the concept of matroids and integral representations. These results illuminate the true meaning of Kron's remark that '*graphs are illegitimate tools for network analysis*' [11].

References

[1] S.H. Birkett and P.H. Roe. The mathematical foundations of bond graphs I. — algebraic theory. *J. Franklin Inst.*, 326:329–350, 1989.

[2] S.H. Birkett and P.H. Roe. The mathematical foundations of bond graphs II. — duality. *J. Franklin Inst.*, 326:691–708, 1989.

[3] S.H. Birkett and P.H. Roe. The mathematical foundations of bond graphs III. — matroid theory. *J. Franklin Inst.*, 327:87–108, 1990.

[4] S.H. Birkett and P.H. Roe. The mathematical foundations of bond graphs IV. — matrix representations and causality. *J. Franklin Inst.*, 327:109–128, 1990.

[5] A.S. Perelson. Bondgraph junction structures. *Trans. ASME J. Dyn. Syst. Meas. Control*, 97:189–195, 1975.

[6] S.H. Birkett and P.H.Roe. The mathematical foundations of bond graphs V. — orientation and orthogonality for directed bondgraphs. *J. Franklin Inst.*, 2002. submitted.

[7] J.R. Ort and H.R. Martens. The properties of bondgraph junction structure matrices. *Trans. ASME J. Dyn. Syst. Meas. Control*, 95:362–367, 1973.

[8] A.S. Perelson. Bondgraph sign conventions. *Trans. ASME J. Dyn. Syst. Meas. Control*, 97:184–188, 1975.

[9] S.H. Birkett. On the special properties of graphic and co-graphic bondgraphs. *J. Franklin Inst.*, 330(4):153–174, 1993.

[10] S.H. Birkett. *Combinatorial Analysis of Dynamical Systems*. PhD thesis, University of Waterloo, Waterloo, Ontario, Canada, 1990.

[11] G. Kron. Graphs as illegitimate models of electrical networks. *Matrix and Tensor Quart.*, 12:1–10, 1961.

Appendix. A Matroid Contraction Theorem

Let M be a matroid on the set S and $T \subseteq S$ with complement $T' = S - T$. The *restriction* of M to T, denoted by $M \mid T$, is the matroid whose circuits are all circuits of M which are contained in T. The restriction $M \mid T$ is said to be obtained from M by *deleting* T' and is also denoted by $M \times T'$. The *contraction* of M to T, denoted by $M \cdot T$, is the matroid whose circuits are the minimal non-null sets of the form $C \cap T'$, where C is a circuit of M. The contraction $M \cdot T$ is said to be obtained from M by *contracting* T' and is also denoted by $M \circ T'$. A matroid is a *minor* of M if it obtained from M by any combination of restrictions and contractions of subsets of S.

Theorem M-1 If M is a matroid on a set S and $T \subseteq S$ then

$$\begin{aligned}(M \times T)^* &= M^* \circ T \\ (M \circ T)^* &= M^* \times T\end{aligned}$$

Figure Captions

Figure 1. Illustration used in proof of Lemma 1.

Figure 2. Causal loop in proof of Theorem 3.

Figure 3. Di-bondgraph with odd causal loop giving the base $\beta = 135$ for the cycle matroid.

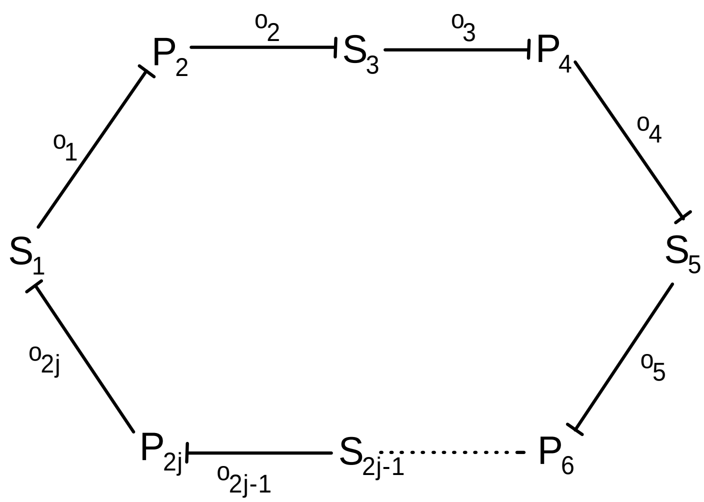
Figure 4. Di-bondgraph and extracted $\bar{B}_{2,4}$ minor.

Figure 5. Bondgraph such that every directed version is regular and has an odd loop.

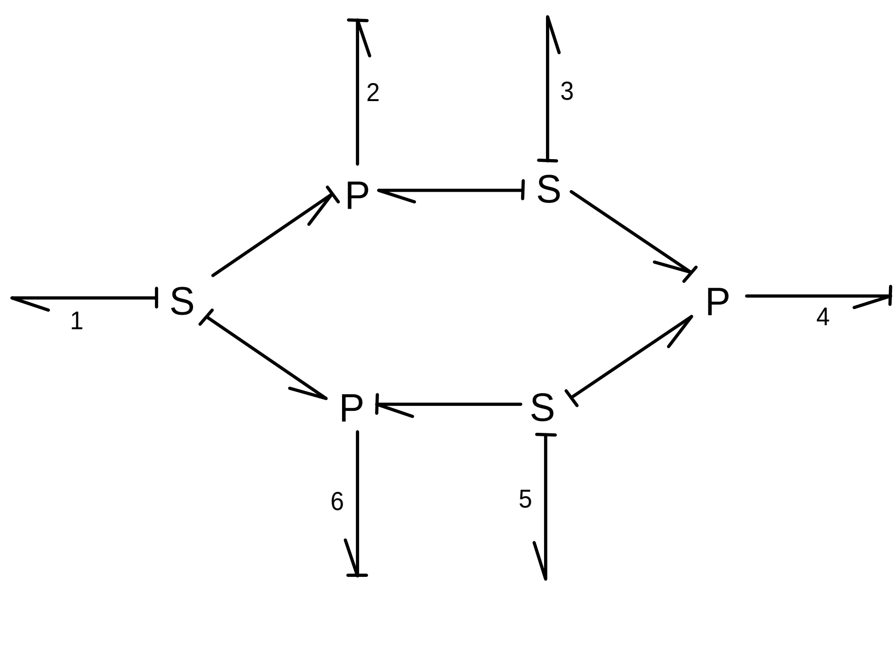
Figure 6. A directed version of the dual Fano bondgraph and extraction of the minor $\bar{B} \times 6 \circ 37$, obtained by removing the central p-junction and the bonds 3 and 6.

$$\xrightarrow[o_1]{} S \xleftarrow[o_2]{} P \xrightarrow[o_3]{}$$

$$\xrightarrow[o_1]{} P \xleftarrow[o_2]{} S \xrightarrow[o_3]{}$$

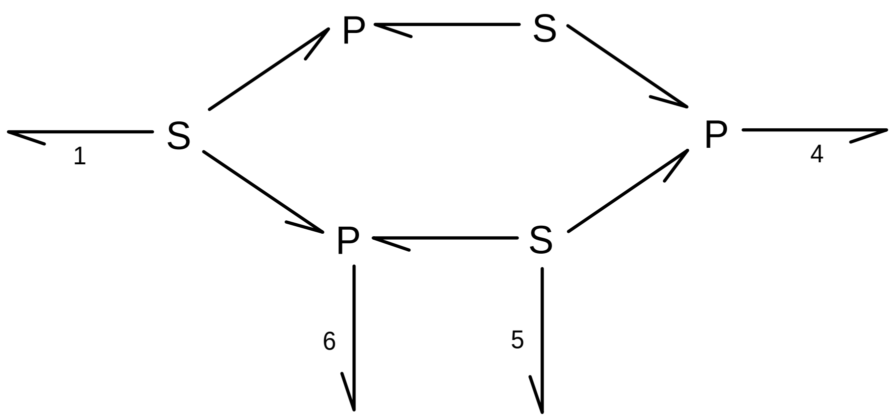


P2
o2
S3
o3
P4
o1
o4
S1
S5
o2j
o5
P2j
S2j-1
P6
o2j-1

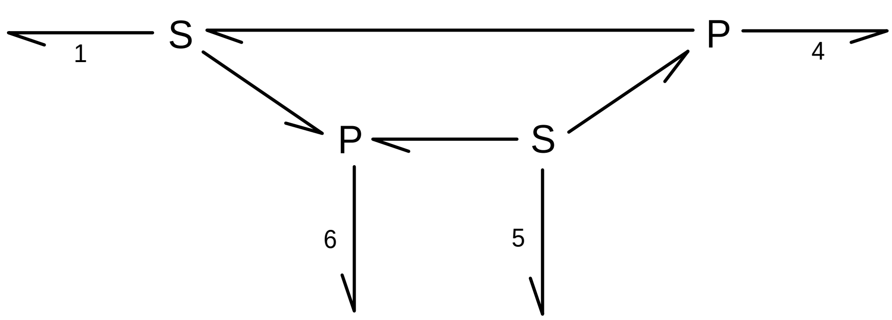


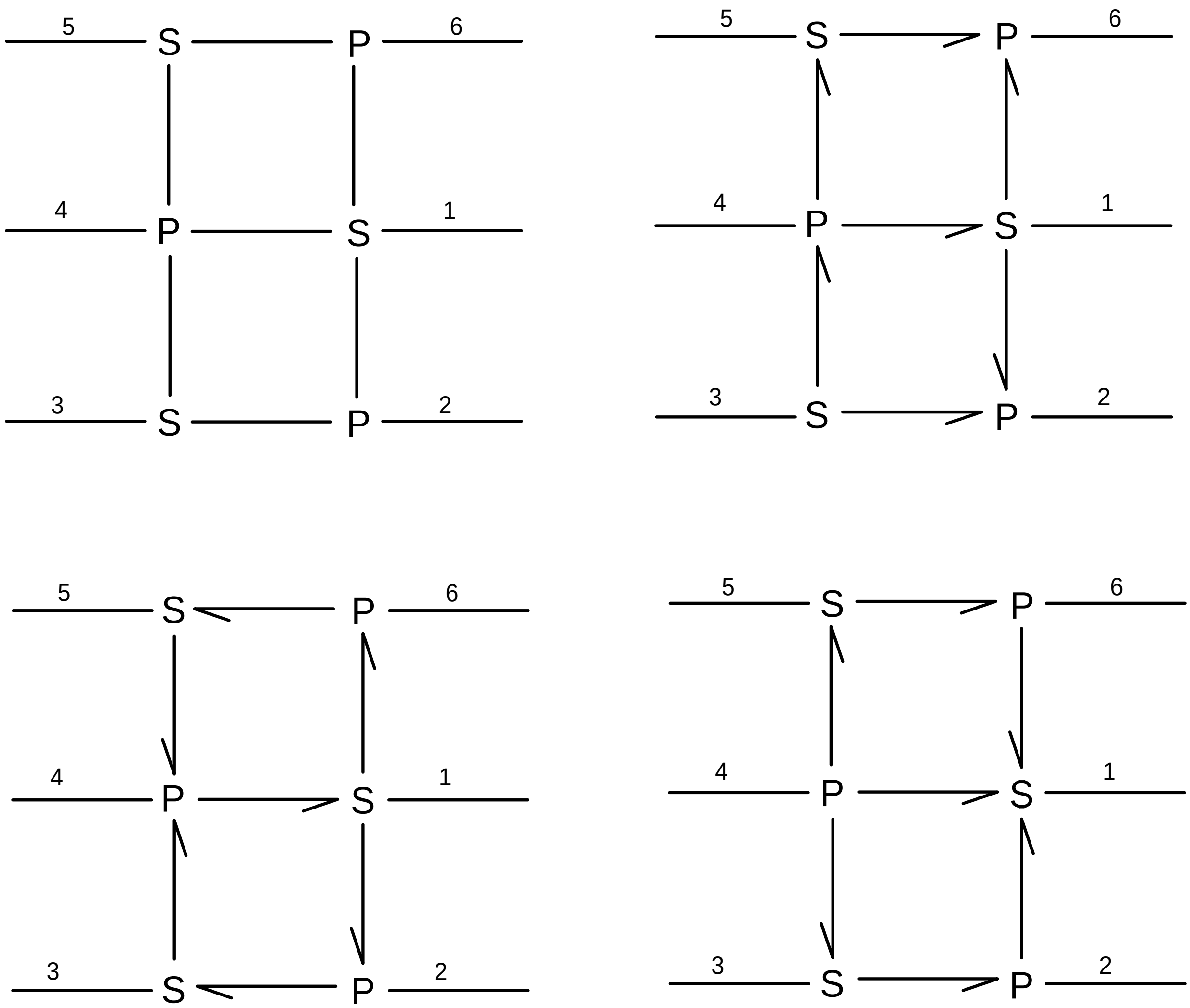
2
3
P
S
S
P
1
4
P
S
6
5

$\bar{B}$

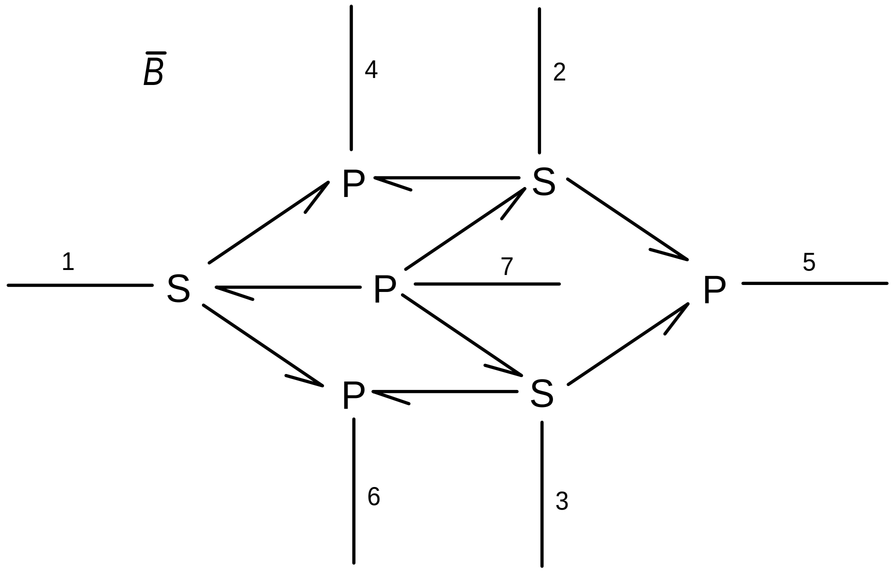


$\bar{B}x2o3$

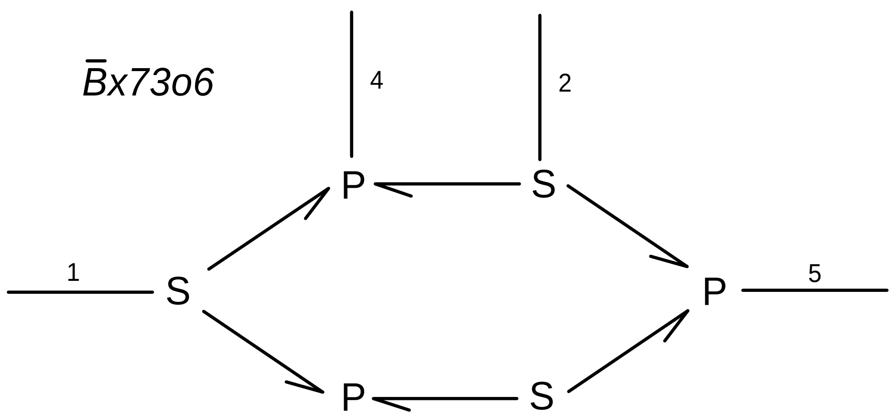



5 S P 6
4 P S 1
3 S P 2
5 S P 6
4 P S 1
3 S P 2
5 S P 6
4 P S 1
3 S P 2
5 S P 6
4 P S 1
3 S P 2

$\bar{B}$



$\bar{B}x73o6$



$\bar{B}x73o6$