The Mathematical Foundations of Bondgraphs V. — Orientation and Orthogonality for Directed Bondgraphs

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Abstract

The presentation of a theory for the mathematical foundations of bondgraphs is continued with an analysis of directed bondgraphs (di-bondgraphs). A di-bondgraph, $\bar{B}$, is formed by adding a half-arrow on each bond of a bondgraph, B, called the underlying bondgraph. There is no *a priori* connection between a bondgraph and its underlying graph and, in general, these represent unrelated combinatorial information. However, an important class of di-bondgraphs, called regular di-bondgraphs, can be defined, for which B and $\bar{B}$ do represent the same structure, called the cycle matroid, and its dual, the co-cycle matroid. A di-bondgraph, $\bar{B}$, is regular if and only if it is orientable, in the sense that the half-arrows induce an orientation on the cycle and co-cycle matroids. Even though non-regular di-bondgraphs are not orientable they can represent the structure of a physical system, and provide a full combinatorial model, for instance for underspecified systems such as the differential gearbox. The intuitive nature of graph-theoretic or network methods has contributed to the common misconception that an oriented combinatorial structure is necessary to represent polarities, but this analysis of di-bondgraphs demonstrates that this is false. The crucial requirement for a combinatorial physical system model is simply a pair of integral representations of a matroid, the cycle matroid, and its dual, the co-cycle matroid, that define the system topology. It is proved that the primal and dual structures represented by a di-bondgraph are always orthogonal (or pseudo-orthogonal if there is an inwardly directed external bond). The precise relationship between regular di-bondgraphs and di-graphs as pictorial representations of combinatorial information is also explored, proving that linear graph and bondgraph system models have no mathematical differences and identical capabilities as modelling tools.

1 Introduction

Two modelling approaches – linear graphs [1, 2, 3, 4] and bondgraphs [5, 6, 7, 8] – are routinely used to represent a spatially discrete physical system, i.e. one which can be described by a finite number of spatially localized physical interactions (components). Superficially,

these two well-established methods seem to rely centrally on quite distinct visual representations of the component interconnections to indicate the relationships and interactions between them. However, only the abstract combinatorial relationships themselves are actually required, and the specific graphical means by which this information is illustrated, whether indirectly through a linear graph, or more directly using a bondgraph junction structure, is irrelevant as far as the physical system model is concerned. Misunderstanding of this point has led to speculation that the two methods have intrinsically different capabilities [9], which is not the case. In fact, bondgraphs and linear graphs have essentially identical capabilities as modelling tools, and any differences are superficial. There may be some pedagogical and/or psychological reasons to prefer one or the other method, but these considerations are not relevant to establishing the technical capabilities of the two methods. These considerations lead to the important practical conclusion that any technique, formulation, computer software, computational algorithm, or model application developed in terms of one methodology will have an identical parallel manifestation in terms of the other method. However, the lack of a rigorous mathematical underpinning for the theoretical aspects of bondgraph methodology has always hampered attempts to establish a definitive proof of this result.

Perceived differences between linear graph and bondgraph approaches may frequently be traced to an over-emphasis on electrical analogues. For example, Paynter and Beaman [10] provide for two choices of system representation, 'either as an equivalent electric circuit or in terms of bondgraphs.' Furthermore, it is stated that bondgraphs have clear advantages in terms of their ability to correctly represent multiple energy domains. These comments are echoed by Lefèvre [9]who suggests that 'any trial to extend networks to all energetic domains must lead to a bondgraph-like representation'. Network models are considered to be 'neither interdisciplinary nor intuitive', because of their focus on electrical analogues, and are limited, for instance in representation of thermodynamic systems, or in accounting for nonlinear kinematic constraints when mechanical system models are generalized 2D or 3D domains. Comments such as these are frequently encountered in the bondgraph literature, even though they are demonstrably incorrect (e.g. see [11]). Sophisticated linear graph techniques such as multi-terminal representations [12, 13, 4]are analogous to the concept of bondgraph multiports; basic coupled components which represent transducers in a linear graph model [13, 2]are no different from the standard bondgraph TF and GY 2-ports [7]. It is a specious argument to dismiss linear graph models simply because 'classical circuits bookkeep electric energy alone' [10]. In fact, the scope of applications based on linear graph modelling is just as wide as those for bondgraphs [14], and, moreover, there is both a considerable degree of overlap and a noticeable lack of interaction between bondgraph and linear graph modelling work, something which is detrimental to the development of both methodologies.

These points may be illustrated by considering specific application areas. For example, a vector-network model was actually developed as early as 1972 [15], and applied to constrained mechanical systems [16, 17, 18], a technique which parallels the vector bondgraph approach to the same problem [19, 20]. In general, multibody dynamics applications have

been developed independently for both bondgraph [21, 22] and linear graph models [23]. Recently, sophisticated and powerful computational techniques using linear graph models have been developed to represent flexible multibody systems for robotic simulation [24], and this again is parallel to similar independent bondgraph applications [25]. Other similar examples can be seen in such applications as probabilistic systems, e.g. a bondgraph extension to probability distribution functions [26] which can be compared to the use of probabilistic variables and system parameters in linear graph models [27]; sensitivity applications have been investigated for many years using linear graphs [28, 29], and this problem is also being investigated separately using bondgraphs [30]; practical simulation of thermodynamic systems is a very well-developed and mature application of linear graph modelling [31, 32, 33], with exactly the same capabilities as those of bondgraph thermodynamic models [34]. As a final example, analysis of control for switched systems has been examined with both bondgraph [35] and linear graph models [36]. In all of the applications cited there has been essentially no interaction between the two groups of researchers.

This situation is unfortunate, not only because it has led to duplication of research efforts, but, more importantly, because it precludes serious interaction and the concomitant benefits which could be derived from pooling results derived by linear graph and bondgraph modellers working together on the same problems. In order to address this issue we undertook to develop a mathematical theory for bondgraphs, which we believe to be important for several reasons:

- Graph-theoretic modelling already has a well-established theoretical mathematical foundation – viz the theory of linear graphs [37, 38] – while bondgraph modelling is based on ad hoc rules.
- The equivalent capabilities of linear graph and bondgraph models can only be established unequivocally when examined in precise mathematical terms.
- A mathematical framework is provided for a coordinate-free combinatorial methodology to model discrete physical systems [39], i.e. one in which the pictorial representation is irrelevant, generalizing the techniques used in both linear graph and bondgraph modelling and providing a means to adapt models easily from one approach to the other.

The focus of this theory was directed initially at examining the capabilities of the bondgraph notation as a representation of combinatorial information in a way which precisely parallels the use of linear graphs in formulating system models. Linear graph techniques make a clear distinction between physical information, i.e. how components behave, and combinatorial information, i.e. how components are interconnected spatially. Consequently, to facilitate the comparative analysis of the two techniques, it is important to express the combinatorial theory for bondgraphs in precisely these same terms. In this context there are four essential requirements: (i) combinatorial relations that define algebraic (spatially-based) constraints between variables in a set X; (ii) a set of dual variables Y and dual algebraic constraints between these variables; (iii) an orthogonality relationship between

the dual structures, ultimately related to a conserved quantity in the physical system; (iv) a means to represent polarities of system variables. Matrix representations of the combinatorial structure (i.e. incidence equations, junction structure equations, etc.) are not required, nor is it necessary to assign causality, i.e. an identification of special variables in the set X (primary variables), which can be used to express algebraically the remaining variables (secondary variables) in X, and similarly for the dual variables in Y. Of course, matrix representations and causality can be useful devices for efficient equation formulation and solution, but these concepts are not an intrinsic aspect of a combinatorial discrete physical system model.

The first part of our theory [40, 41, 42, 43, 44] examines the concept of *(non-directed) bondgraph*, an abstraction of the familiar 'bondgraph junction structure without power half-arrows' [45]. This theory should elucidate the first three of the combinatorial requirements listed above by establishing the one-to-one relationship that exists between non-directed bondgraphs and dual pairs of binary matroids. Moreover, a precise connection with linear graphs is provided by the concept of a graphic bondgraph, i.e. one for which the associated matroid has a linear graph representation; dually, co-graphic matroids and bondgraphs are such that their dual matroid can be associated with a graph [44]. The class of structures represented by a non-directed bondgraph includes those which can be represented by either a graph or dual graph, called regular matroids, and also those which have neither graph nor dual graph, called non-regular matroids (e.g. the Fano matroid). These same definitions can be applied to the bondgraph representations of these matroids, dividing the class of non-directed bondgraphs into regular and non-regular. The latter are not relevant to physical system models. The conventional bondgraph procedure used to define causality by adding a causal stroke to each bond is a means to select a base for the bondgraph matroid, and is analogous to choosing a spanning tree for a linear graph. The important property of orthogonality between the dual matroid (bondgraph) structures was also established in [41].

In order to complete the analysis of bondgraphs as representations of combinatorial information for physical system models, it is still necessary to establish how a bondgraph model satisfies the fourth requirement above, viz. the representation of polarities for the variables. This question is addressed in this article and a subsequent one [46]by examining the combinatorial properties of directed bondgraphs (di-bondgraphs), which are formed by assigning a direction to the bonds of a (non-directed) bondgraph, conventionally indicated on the bondgraph model diagram by a half-arrow on each bond. A di-bondgraph can be associated with a dual pair of integral matroids. In particular, a di-bondgraph is an explicit visual device from which an integral chain group can be constructed, and this provides the integral represesention of the associated matroid. In general, no *a priori* relationship should be anticipated between a directed and non-directed version of the same bondgraph, however it is possible to establish a general result which determines when this is so, in which case the directed version provides an orientation of the matroid common to both. The results established previously for binary matroids and bondgraphs [42, 43], for instance orthogonality and base selection via causal strokes, must be re-examined in the context

of di-bondgraphs and their integral matroids. The analysis is developed in an entirely coordinate-free approach, however matrix representations of the matroid and chain group structures are also explored, establishing a link to the conventional incidence and junction structure matrices of linear graph and bondgraph methods.

In the decade since the publication of our articles on the combinatorial foundations of non-directed bondgraphs various theoretical results have appeared in the bondgraph literature. Most of these focus on specific technical issues important for the development of bondgraph models, for example a study of the solvability of junction structures [47], an examination of the connections between linear graph trees and bondgraphs [48], a treatment of algebraic loops for multibond graphs [49], or the presentation of a method to link a transfer matrix to a bondgraph model [50]. Although these types of problems do also address the combinatorial aspects of bondgraph modelling methodology, for the most part the issues also involve the physical aspects of the model. A series of three articles [51, 52, 53] presents a combinatorial theory for bondgraphs, however the focus of that work is considerably changed by the decision to include physical components, and, in particular, the TF and GY 2-port elements, in the analysis. As discussed above, our goal has always been to provide a means to highlight the similarities between bond graph and linear graph modelling, rather than develop a complex theory for its own sake. In order to achieve this practical objective, it is essential to restrict the combinatorial basis to avoid physical components, even though this may be an interesting mathematical problem itself. In linear graph modelling a coupled component is an idealized physical relationship and needs to be considered as extraneous to the combinatorial structure of the model rather than part of it. This is why, unlike most theoretical bondgraph work, we have limited our combinatorial theory to junction structures, and have excluded coupled components. The different focus of these three articles makes it difficult to compare their results with our theory, and we will not attempt to establish connections, although there is clearly some overlap in the concepts and results.

Since the publication of our original articles, the connections between bondgraphs and matroids have also been examined by several authors. Reibiger and Loose [54] use matroids to provide an axiomatic link between bondgraph terminology and classical network theory in terms of graphoids and Minty networks [55]. Matroid concepts are applied in [56, 57]to develop a practical technique for assigning causality in a bondgraph. Finally, it should be noted that some additional special results related to graphic and co-graphic bondgraphs were presented in [44]. It is shown, for instance, how to construct for a physical system model canonical bondgraphs which allow alternative formulations analogous to nodal and mesh techniques for linear graph models.

A brief summary of the essential aspects of the mathematical theory of matroids and integral chain groups is provided in Appendix A. Some important mathematical background material was also summarized in [42, 43]; for a more complete treatment the reader is referred to [37]. The definitions, notation and theory of the first four papers in this series [40, 41, 42, 43] are used freely herein. Use of the convenient juxtaposition notation for sets of bonds will be continued: for example '1256' denotes the set with elements 1, 2, 5 and 6. This notation simplifies that for a set whose elements are sets themselves. So, for

instance, a collection of circuits written conveniently as $\{12, 134, 45\}$ using juxtaposition would be written $\{\{12\}, \{134\}, \{45\}\}$ using conventional notation. The symbol $\sharp S$ denotes the number of elements of the set S, called the *cardinality* of S.

In bondgraph physical system models junctions are conventionally notated with the symbols '0' and '1'. However, in the theoretical combinatorial context being developed here numerical junction symbols are quite confusing, becuase much of this presentation involves sets of bonds labelled numerically. Therefore, simply to avoid undue confusion, junction labels throughout this series have been denoted with alphabetic symbols using the letters 'p' and 's'. No special significance should be attributed to these arbitrary choices.

2 Directed Bondgraph Matroids

A *non-directed bondgraph* B can be defined formally [40] as consisting of: (i) an *underlying graph* which has no self-loops and a partition on the vertex set (*junctions*) denoted by the symbols 'p' and 's; and (ii) an additional set of edges E_e (*external bonds*), each of which is associated with a single junction. The edges of the underlying graph are called *internal bonds* (E_i) and junctions that have no incident external bond are called *internal junctions*. A *degenerate junction* is one with precisely two incident bonds. This definition is consistent with the conventional pictorial representation of a bondgraph junction structure [45].

A *directed bondgraph (di-bondgraph)* is a bondgraph in which each bond has been assigned a direction, indicated on the diagram by a half-arrow added to the end of each bond. A di-bondgraph which is defined by assigning directions to the bonds of a bondgraph B is denoted by $\bar{B}$. The bondgraph B is called the *underlying bondgraph* of $\bar{B}$. A di-bondgraph is *simple* if its underlying bondgraph is simple.

Let $\bar{B}$ be a di-bondgraph with bond set E, the set of all bonds of $\bar{B}$, both internal and external. Each s-junction of $\bar{B}$ defines an integral chain on E, f_s, called an *s-chain*, with support the set of all the bonds incident on the junction [the mathematical concepts used here are summarized in Appendix A]. If $b \in \mathrm{supp}(f_s)$ define $f_s(b) = 1$ if b is directed out of the junction and $f_s(b) = -1$ if b is directed into the junction. Dually, each pair of bonds incident on a p-junction of $\bar{B}$ defines an integral chain on E, called a *p-chain*, with those two bonds as support. The directions of these two bonds are disregarded and f_p is defined arbitrarily to be 1 for one of the bonds and -1 for the other bond. The s-chains and p-chains of $\bar{B}$ are called *junction chains*.

Let $\bar{B}$ be a di-bondgraph with bond set E. The set of all junction chains of $\bar{B}$ generates an integral chain group on E, called the *junction chain group* of $\bar{B}$ and denoted by $N_s(\bar{B})$. The *cycle chain group* of $\bar{B}$, denoted by $N_{cy}(\bar{B})$, is the restriction of the junction chain group to the external bond set of $\bar{B}$. The chains of N_{cy} are called *cycle chains*.

A *minimal generating set* of junction chains of a di-bondgraph is a set of junction chains which generates the junction chain group and such that no proper subset of its chains does so. A *minimal generating set* of cycle chains of $\bar{B}$ is a set of chains which generates $N_{cy}(\bar{B})$ and such that no proper subset of its chains does so.

No s-chain may be removed from a generating set of junction chains, however not all

of the p-chains are required to generate the junction chain group. For a p-junction with k incident bonds all of the p-chains for that junction will be generated by a collection of any $k-1$ p-chains such that every incident bond appears at least once.

The *cycle matroid* of the di-bondgraph $\bar{B}$, denoted by $M(\bar{B})$, is the matroid of the cycle chain group of $\bar{B}$, $M(N_{cy}(\bar{B}))$. Equivalently the cycle matroid of $\bar{B}$ is the restriction to E_e, the set of external bonds of $\bar{B}$, of the matroid of the junction chain group of $\bar{B}$.

The *dual di-bondgraph*, $\bar{B}^*$, of a di-bondgraph $\bar{B}$ is the di-bondgraph whose underlying bondgraph is the dual bondgraph of B, B^*, and with the bonds directed the same as those of $\bar{B}$. The *dual junction chains* of $\bar{B}$ are the junction chains of the dual $\bar{B}^*$. The *s^*-chains* of $\bar{B}$ are the p-chains of $\bar{B}^*$ and the *p^*-chains* of $\bar{B}$ are the s-chains of $\bar{B}^*$. The *dual junction chain group* of $\bar{B}$, denoted $N_p(\bar{B})$, is the junction chain group of $\bar{B}^*$. The *co-cycle chain group* of $\bar{B}$, denoted $N_{co}(\bar{B})$, is the cycle chain group of $\bar{B}^*$. The chains of $N_{co}(\bar{B})$ are called *co-cycle chains* of $\bar{B}$. The *co-cycle matroid* of $\bar{B}$, $M^*(\bar{B})$, is the matroid of the co-cycle chain group, the cycle matroid of $\bar{B}^*$, *i.e.* $M^*(\bar{B}) = M(\bar{B}^*)$.

Integral chains, in general, will be denoted with a vector whose k coordinates represent the values on the set $12\ldots k$ on which the chain is defined. For instance $(1,-2,0,-1)$ denotes a chain with respective values 1,−2,0 and −1 on the set 1234. A simplified notation will be used for a primitive integral chain, one which takes only the values −1,0 and 1. Such a chain will be denoted by specifying its support set (using, as always, the juxtaposition notation for this) and placing a bar over any element on which the chain is defined to be −1. For instance $5\bar{2}3\bar{o_1}$ denotes the chain which has the value 1 on elements 5 and 3, the value −1 on elements 2 and o_1 and the value 0 on any other element.

Example 1 Consider the di-bondgraph $\bar{B}$ shown in Figure 1. A minimal generating set of junction chains for the junction chain group consists of the s-chains $s_1 = 12o_1$ and $s_2 = 56\bar{o_2}$ and the three p-chains $p_1 = 3\bar{o_1}$, $p_2 = 3\bar{o_2}$ and $p_3 = 3\bar{4}$. The three chains:

$$x_1 = 123 = s_1 + p_1$$
$$x_2 = \bar{3}56 = s_2 - p_2$$
$$x_3 = 3\bar{4} = p_3$$

are a minimal generating set for the cycle chain group:

	1	2	3	4	5	6	supp
x_1	1	1	1	0	0	0	123
x_2	0	0	−1	0	1	1	356
x_3	0	0	1	−1	0	0	34
$x_1 + x_2$	1	1	0	0	1	1	1256
$x_1 - x_3$	1	1	0	1	0	0	124
$x_2 + x_3$	0	0	0	−1	1	1	456
$x_1 + x_2 + x_3$	1	1	1	−1	1	1	123456

The supports of the elementary chains:

$$\{123, 356, 34, 1256, 124, 456\}$$

are the circuits of the cycle matroid, $M(\bar{B})$, which is the same as the cycle matroid of the underlying bondgraph B.

A minimal generating set of dual junction chains for the dual junction chain group consists of the p^*-chain $p_1^* = 34\bar{o_1}o_2$ and the four s^*-chains $s_1^* = 1\bar{o_1}$, $s_2^* = 1\bar{2}$, $s_3^* = 5\bar{o_2}$ and $s_4^* = 5\bar{6}$. The three chains:

$$y_1 = \bar{1}345 = p_1^* - s_1^* + s_3^*$$
$$y_2 = 1\bar{2} = s_2^*$$
$$y_3 = 5\bar{6} = s_4^*$$

are a minimal generating set for the co-cycle chain group:

	1	2	3	4	5	6	supp
y_1	−1	0	1	1	1	0	1345
y_2	1	−1	0	0	0	0	12
y_3	0	0	0	0	1	−1	56
$y_1 + y_2$	0	−1	1	1	1	0	2345
$y_1 - y_3$	−1	0	1	1	0	1	1346
$y_2 + y_3$	1	−1	0	0	1	−1	1256
$y_1 + y_2 - y_3$	0	−1	1	1	0	1	2346

The supports of the elementary chains:

$$\{1345, 12, 56, 2345, 1346, 2346\}$$

are the circuits of the co-cycle matroid, $M^*(\bar{B})$, which is the same as the co-cycle matroid of the underlying bondgraph B.

The co-cycle chains provide an integral representation of the cycle matroid of $\bar{B}$:

1	2	3	4	5	6
−1	0	1	1	1	0
1	−1	0	0	0	0
0	0	0	0	1	−1

and the cycle chains provide an integral representation of the co-cycle matroid of $\bar{B}$:

1	2	3	4	5	6
1	1	1	0	0	0
0	0	−1	0	1	1
0	0	1	−1	0	0

Alternatively if the chains of the cycle and co-cycle chain groups are written as the rows of matrices D and D^* respectively, since every chain is primitive, D is a signed circuit matrix and D^* is a signed co-circuit matrix for M. Furthermore the rows of D are orthogonal to the rows of D^* and so the di-bondgraph provides an orientation of the cycle matroid of the underlying bondgraph.

Example 2 Consider the bondgraph $\bar{B}$ shown in Figure 2. Using the junction chains it can be seen that the two chains $x_1 = 1\bar{2}\bar{4}$ and $x_2 = \bar{2}34$ generate the cycle chain group:

	1	2	3	4	
x_1	1	-1	0	-1	124
x_2	0	-1	1	1	234
$x_1 - x_2$	1	0	-1	-2	134
$x_1 + x_2$	1	-2	1	0	123

The chains $x_1 - x_2 = (1, 0, -1, -2)$ and $x_1 + x_2 = (1, -2, 1, 0)$ are elementary chains for $N_{cy}(\bar{B})$. The cycle chain group is non-regular, since, for instance, there is no primitive chain with the same support as the elementary chain $(1, -2, 1, 0)$. The circuits of the cycle matroid of $\bar{B}$ are the supports of the elementary chains of $N_{cy}(\bar{B})$:

$$\{124, 234, 134, 123\}.$$

The matroid with these circuits is the uniform matroid $U_{2,4}$, which is the smallest non-binary non-regular matroid. Clearly, in this example, the cycle matroid of the di-bondgraph has no connection with the cycle matroid of the underlying bondgraph.

A di-bondgraph with $U_{2,4}$ as cycle matroid is called the *uniform di-bondgraph of rank 2 on 4 bonds* and is denoted by $\bar{B}_{2,4}$.

Two di-bondgraphs are *topologically equivalent* if they have isomorphic cycle matroids. They are *equivalent* if they also have precisely the same cycle chain group and the same co-cycle chain group. Topologically equivalent di-bondgraphs are not necessarily equivalent. This can occur because either the cycle chain group, the co-cycle chain group or both of these chain groups are different.

Example 3 Reversing the direction on either of the internal bonds of the di-bondgraph in Figure 1 (Example 1) gives a topologically equivalent di-bondgraph, since neither the cycle nor co-cycle matroid is altered. The new di-bondgraph is not equivalent to the original one, though, since its cycle and co-cycle chain groups are different from those of the original di-bondgraph. Reversing the direction on one of the external bonds, b, changes the cycle chain group, if b is on an s-junction, or the co-cycle chain group, if b is on a p-junction. In either case the new di-bondgraph is topologically equivalent but not equivalent to the original one.

Reversing the direction on one of the internal bonds in the di-bondgraph of Figure 2 (Example 2) gives a di-bondgraph which is not topologically equivalent to the original one.

Example 1 and Example 2 illustrate two possibilities which can occur in the relation between a di-bondgraph and its underlying bondgraph. In the first case the di-bondgraph and the bondgraph have the same matroids and the di-bondgraph provides an orientation of the cycle matroid of the underlying bondgraph. In the second case the cycle matroid of the di-bondgraph is not regular and hence non-orientable also. In this case the matroids of

the di-bondgraph and its underlying bondgraph are not related. Both of these situations can arise in the use of di-bondgraphs in physical system modelling. However there is a lack of understanding of the combinatorial and physical validity of di-bondgraph models with structure of the second type, which can occur, for instance, when there is a so-called odd-loop of internal bonds [58]. The use of these odd-loops in a di-bondgraph model of this type is mathematically correct and may be used when it is appropriate to model the structure of the physical system with a non-regular matroid, such as $U_{2,4}$. After establishing these general results their implications are discussed in more detail below.

3 Regularity

A bondgraph or di-bondgraph is *regular* if it has a regular cycle chain group and *matroid-regular* (*m-regular*) if it has a regular cycle matroid. A bondgraph or di-bondgraph which is not regular is called *non-regular*.

Theorem 1 A regular di-bondgraph is m-regular.

Proof: The matroid of a regular chain group is regular, by definition, and the cycle matroid of a di-bondgraph is the matroid of its cycle chain group.

Example 4 Consider the di-bondgraph $\bar{B}$ of Figure 4. The junction chains can be used to give generating chains for the junction chain group restricted to the set $1234o_1$, the set of external bonds and one internal bond labelled o_1:

	1	2	3	4	o_1	supp
x_1	1	0	0	1	1	$14o_1$
x_2	-1	1	0	0	1	$12o_1$
x_3	0	0	-1	1	1	$34o_1$
$x_1 - x_2$	2	-1	0	1	0	124
$x_1 - x_3$	1	0	1	0	0	13
$x_2 - x_3$	-1	1	1	-1	0	1234
$x_1 + x_2 - 2x_3$	0	1	2	-1	0	234

Deleting the internal bond o_1 restricts to the set 1234. The circuits of $M(\bar{B})$ correspond to the elementary chains which do not involve o_1. There are three of these: $\{124, 13, 234\}$, which are easily seen to be the circuits of a binary matroid. This cycle chain group is non-regular (see Example 10), but the cycle matroid of this di-bondgraph is regular. So $\bar{B}$ is m-regular but not regular.

If an additional external bond is included in $\bar{B}$, incident on the same p-junction as o_1, the new cycle chain group is the same as the chain group given above for the set $1234o_1$. Both the new cycle chain group and its new cycle matroid M' are non-regular. The cycle matroid $M(\bar{B})$ is a regular minor of M', obtained by deleting the element corresponding to the extra external bond.

Example 4 shows that the converse of Theorem 1 is false: an m-regular di-bondgraph may be non-regular. This is a consequence of the fact that a regular matroid can be the matroid of a non-regular chain group. Such pathological examples are regular minors of non-regular matroids and arise as the cycle matroids of di-bondgraphs for which the matroid of the junction chain group is non-regular (*i.e.* including both external and internal bonds). When restricted to the external bond set the matroid becomes regular, but the defining chain group remains non-regular. The non-regular di-bondgraphs which are also not m-regular form a more important and interesting class of di-bondgraphs.

In [42] it is shown how to associate a binary matroid M with a bondgraph B. The bondgraph shows, indirectly, the chains of a binary chain group of which M is the matroid. A di-bondgraph $\bar{B}$ provides a representation of an integral matroid M, by giving, indirectly, the chains of an integral chain group of which M is the matroid. These results are characterized, combined with that of Theorem M-10 [Appendix A], in the diagram of Figure 3. Regular matroids, those which are both integral and binary, are associated either with a regular bondgraph, which gives a binary representation, or with an m-regular di-bondgraph, which gives an integral representation. The two classes of non-regular matroids are characterized by excluded minors, as shown in the diagram. Note that $U_{2,4}$ is self-dual and so no dual matroid appears in this part of the diagram.

Di-bondgraphs may be simplified by contractions of bonds and junctions, but these must be done carefully so as not to change the integral chain group represented. Valid *elemental contractions* of a di-bondgraph $\bar{B}$ are defined as follows: (i) An internal bond joining any two junctions of the same type is removed from $\bar{B}$ and the junctions identified into one junction of the same type as the original two without changing the directions on any of the incident bonds; (ii) A degenerate internal junction with oppositely directed incident bonds is changed to a junction of opposite type; or (iii) Two adjacent degenerate junctions, one with similarly directed bonds and the other with oppositely directed bonds, are changed to junctions of the opposite type. A di-bondgraph is *proper* if it is not possible to contract any internal bond or any degenerate internal junction. The *proper contraction* of a di-bondgraph is the di-bondgraph remaining after all valid elemental contractions have been performed.

Theorem 2 Let $\bar{B}$ be a di-bondgraph. A di-bondgraph equivalent to $\bar{B}$ is obtained by any combination of contractions of internal bonds or junctions with two incident bonds. In particular, the proper contraction of $\bar{B}$ is a proper di-bondgraph equivalent to $\bar{B}$.

Proof: The contraction of an internal bond joining junctions of the same type produces a di-bondgraph which represents the sum of two junction chains of $\bar{B}$ as a single junction chain. Changing the type of a junction which has two oppositely directed (internal) bonds does not affect the junction chain on that junction. Finally, the two pairs of adjacent junctions shown in Figure 5 are equivalent since each represents the chain $o_1 o_3$, the first being the sum $o_1 o_2 + \bar{o_2} o_3$ and the second being the sum $o_1 \bar{o_2} + o_2 o_3$.

Theorem 3 Let $\bar{B}$ be a simple di-bondgraph with underlying bondgraph B. Then $\bar{B}$ is regular if and only if the cycle matroids of B and $\bar{B}$ are the same, *i.e.* $M(B) = M(\bar{B})$.

Proof: If $\bar{B}$ is regular then its cycle matroid is the matroid of the regular cycle chain group $N_{cy} = N_{cy}(\bar{B})$. Each circuit of $M(\bar{B})$ corresponds to an elementary chain of N_{cy} and, since N_{cy} is regular, there is also a primitive chain corresponding to this circuit. So the primitive chains of N_{cy} can be put into one to one correspondence with the circuits of $M(\bar{B})$. Now an arbitrary primitive chain $x \in N_{cy}$ can be expressed as a linear combination of junction chains of $\bar{B}$, all of which are primitive since $\bar{B}$ is simple. Because supp(x) contains no internal bond each internal bond in the linear combination must occur in an even number of junction chains. This follows from the fact that the non-zero coefficients in the junction chains must be ± 1 and every internal bond in supp(x) has a coefficient of zero. Similarly each external bond in supp(x) must have a coefficient of ± 1 and so it must occur in an odd number of junction chains in the linear combination. Therefore, removing all the negative signs in the linear combination, gives a linear combination of elementary junction vectors of B, a cycle of B, which is the same as supp(x). The cycles of B are the circuits of $M(B)$ and so there is a one to one correspondence between the circuits of $M(\bar{B})$ and the circuits of $M(B)$.

Conversely, if $M(\bar{B}) = M(B) = M$, then $\bar{B}$ provides an integral representation of M and B provides a binary representation of M. Thus M is binary and integral and so, by Theorem M-10, M is regular, *i.e.* $\bar{B}$ is m-regular. If $\bar{B}$ is non-regular then $N_{cy}(\bar{B})$ is a non-regular chain group and, as shown in Example 4, $\bar{B}$ can be augmented by adding an extra external bond on some of the p-junctions to form a di-bondgraph $\bar{B}'$ which is not m-regular. Now, if B' denotes the corresponding bondgraph, augmented in the same way, then $M(\bar{B}') = M(B')$. But, by Theorem M-10, a non-regular integral matroid cannot be binary and thus cannot be the matroid $M(B')$ of a non-directed bondgraph B'. This contradiction implies that the di-bondgraph $\bar{B}$ must be regular.

The *rank* of a di-bondgraph $\bar{B}$, denoted by $\rho(\bar{B})$, is the rank of $M(\bar{B})$. Dually the *co-rank* of $\bar{B}$, denoted by $\rho^*(\bar{B})$, is the rank of $M^*(\bar{B})$.

Theorem 4 Let $\bar{B}$ be a regular simple proper di-bondgraph. The cycle and co-cycle matroids of $\bar{B}$ are dual matroids. Any base set β corresponding to a base colouring of B, is a base of $M(\bar{B})$ and

$$\begin{aligned} \rho(\bar{B}) &= p - s + e_s \\ \rho^*(\bar{B}) &= s - p + e_p \end{aligned}$$

give the rank and co-rank of $\bar{B}$.

Proof: By Theorem 3 the cycle matroids of a regular simple di-bondgraph and its underlying bondgraph are the same. By duality, the co-cycle matroids must also be the same. Hence the cycle and co-cycle matroids are dual matroids. Also the bases of $M(B)$ which are determined by base colourings of B are also bases of $M(\bar{B})$ and $\rho(\bar{B}) = \rho(B)$. The formulas for the rank and co-rank of $\bar{B}$ follow from *Theorem XII* of [43].

4 Orthogonality

An *orthogonal* di-bondgraph is one in which every external bond is directed out of the junction.

Lemma 1 Let $\bar{B}$ be a simple di-bondgraph in which every internal bond joins junctions of opposite type. If s denotes any s-chain and s^* denotes an s^*-chain then $\langle s \mid s^* \rangle$ can have only the values -2, 0 or 2. The non-zero values occur only when the chains are formed on the same s-junction. Dually, if p denotes a p-chain and p^* denotes an p^*-chain then $\langle p \mid p^* \rangle$ can have only the values -2, 0 or 2 and the non-zero values occur only when the chains are formed on the same p-junction.

Proof: From *Lemma 2* of [41] an s- and s^*-chain have two common bonds if and only if they are formed on the same junction and otherwise no common bond. The inner product $\langle s \mid s^* \rangle$ is zero when both bonds in the s^*-chain are directed the same, for they will have the same sign in the s-chain and opposite signs in the s^*-chain. The non-zero values -2 or 2 occur when the two bonds in the s^*-chain are directed oppositely, for in this case they will have opposite signs in both of the chains.

The following theorem is the critical orthogonality property which is essential to the application of di-bondgraphs to physical system modelling.

Theorem 5 The cycle chain group of a simple orthogonal di-bondgraph is orthogonal to the co-cycle chain group.

Proof: Without loss of generality assume that $\bar{B}$ is proper, for otherwise consider the equivalent proper contraction of $\bar{B}$. Let $x \in N_{cy}(\bar{B})$ and $y \in N_{co}(\bar{B})$ be arbitrary chains. The cycle chain x has a decomposition as a sum of s- and p-chains and the co-cycle chain y has a decomposition as a sum of s^*- and p^*-chains. Assume that these decompositions are chosen so as to involve the minimum number of junction chains and that no junction chain appears more than once.

The result of the theorem is proved by appealing to the linearity of the inner product and showing that the sum of all contributions from inner products of junction chains with dual junction chains is zero. Arrange the dual junction chains of the decomposition of y as the columns of an array, A, and the junction chains of x as the rows. At position jk of A the value a_{jk} is defined to be the inner product of the junction chain in row j with the dual junction chain in column k.

Consider the non-zero values of a_{jk}. These occur only when there is at least one common bond at position jk. The contributions from internal bonds can be included by 'stamping' four values as follows. One such stamp is defined for each pair of adjacent p- and s-junctions and the values for the only two possible stamps are:

	s^*	p^*
s	1	-1
p	-1	1

	s^*	p^*
s	-1	1
p	1	-1

These values follow because an internal bond which occurs in both decompositions must be common to an s-chain and a p-chain on adjacent junctions and also common to the s^*-chain and a p^*-chain on those junctions. Furthermore the signs in the decomposition of x must be opposite and the signs in the decomposition of y must be opposite since the support of a cycle or co-cycle chain includes only external bonds and so the internal bonds must cancel.

Now the off-diagonal terms in these stamps must be ± 1 and the diagonal terms must be 0 or ± 2, by Lemma 1. Therefore there must be one further contribution of value ± 1 at the position of each diagonal entry of each stamp and no further contributions at positions of off-diagonal entries.

First suppose that no two stamps intersect. In this case there must be two additional entries of value ± 1 in each stamp, one at each of the diagonal positions, and, because no stamps intersect, these must occur because of external bonds. Consider one of the stamps, corresponding to a pair of adjacent junctions and let the external bond common to both the s- and s^*-chains be labelled 1 and the external bond common to both the p- and p^*-chains be labelled 2. The di-bondgraph segment for this situation is shown in Figure 6. Any other bonds incident on either of the junctions are irrelevant to the stamp under consideration. The additional stamp entries are found by examining the junction chains for this segment. The only two possible complete stamps for this segment are shown below:

	s^*		p^*	
s	1	$+1$	-1	
p		-1	1	-1

	s^*		p^*	
s	-1	-1	1	
p		1	-1	$+1$

If the internal bond in the segment is directed toward the p-junction, duality can be used to obtain the following two complete stamps:

	s^*		p^*	
s	-1	$+1$	-1	
p		-1	1	$+1$

	s^*		p^*	
s	1	-1	1	
p		1	-1	-1

In deriving these complete stamps the fact that all external bonds are directed away from the junction is crucial. For the case of non-intersecting stamps it is clear that the net contribution from each stamp is zero.

Consider a pair of intersecting stamps, corresponding to a sequence of three adjacent junctions. This pair of stamps can be regarded as a pair of non-intersecting stamps by introducing a 'dummy' external bond, say b, on the middle junction. If the middle junction is a p-junction re-write the p-chain $o_1\bar{o}_2$ as $o_1\bar{o}_2 = o_1\bar{b} + b\bar{o}_2$ which makes no change to x. The p^*-chain has one additional bond in it, b, which means that the dual chain y has been changed by the addition of the bond b. But, since x does not contain the bond b, there will be no change to the inner product $\langle x \mid y \rangle$. Consideration of the stamp array shows that the row on which the pair of stamps intersected previously has now been split into two rows by the introduction of the extra bond b. Thus there will be two stamps of the non-intersecting type shown above and so no net contribution. Duality gives the same result for the case of

a middle s-junction, in which case the s^* column will be split into two columns to provide non-intersecting stamps.

Every pair of intersecting stamps can be treated in the manner described above and so the net contribution from all stamps is zero, *i.e.* $\langle x \mid y \rangle = 0$.

Example 5 Consider the di-bondgraph shown in Figure 7. The proof of Theorem 5 is illustrated by constructing the array A for the inner product of the cycle chain $x = 34567$ and the co-cycle chain $y = 1\bar{3}6\bar{8}$. In terms of elementary junction chains,

$$x = 34\bar{o}_2 + o_2\bar{o}_3 + o_35\bar{o}_4 + o_4\bar{o}_5 + o_567y = 1\bar{o}_1 + o_2\bar{3} + o_1\bar{o}_2o_3 + \bar{o}_3o_4 + \bar{o}_4o_5\bar{8} + \bar{o}_56$$

The four stamps in the inner product of these two chains are:

		s^*		p^*	
		$o_2\bar{3}$		$o_1\bar{o}_2o_3$	
s	$34\bar{o}_2$	-1	-1	$+1$	
p	$o_2\bar{o}_3$		$+1$	-1	-1

		p^*		s^*	
		$o_1\bar{o}_2o_3$		$\bar{o}_3o_4$	
p	$o_2\bar{o}_3$	-1	-1	$+1$	
s	$o_35\bar{o}_4$		$+1$	-1	-1

		s^*		p^*	
		$\bar{o}_3o_4$		$\bar{o}_4o_5\bar{8}$	
s	$o_35\bar{o}_4$	-1	-1	$+1$	
p	$o_4\bar{o}_5$		$+1$	-1	-1

		p^*		s^*	
		$\bar{o}_4o_5\bar{8}$		$\bar{o}_56$	
p	$o_4\bar{o}_5$	-1	-1	$+1$	
s	o_567		$+1$	-1	$+1$

When collected to form the array A, the four intersecting stamps combine to produce a net contribution of zero:

		s^*	s^*		p^*		s^*		p^*		s^*	
		$1\bar{o}_1$	$o_2\bar{3}$		$o_1\bar{o}_2o_3$		$\bar{o}_3o_4$		$\bar{o}_4o_5\bar{8}$		$\bar{o}_56$	
s	$34\bar{o}_2$		-1	-1	1							
p	$o_2\bar{o}_3$			1	-1	-1	1					
s	$o_35\bar{o}_4$					1	-1	-1	1			
p	$o_4\bar{o}_5$							1	-1	-1	1	
s	o_567									1	-1	$+1$

Example 6 The proof of Theorem 5 is illustrated for the non-regular di-bondgraph $\bar{B}_{2,4}$ shown in Figure 8. The array A for the cycle chain $x = (1, 1, 0, 1)$ and the co-cycle chain $y = (-2, 1, 0, 1)$ is constructed below:

		p^*		s^*		s^*	s^*		p^*		s^*	
		$2\bar{o}_1o_2$		$\bar{1}o_1$		$3\bar{o}_2$	$\bar{1}o_4$		$4\bar{o}_3\bar{o}_4$		$o_3\bar{3}$	
p	$2\bar{o}_1$	$+1$	$+1$	-1								
s	$1o_1o_4$		-1	1	-1		-1	$+1$	-1			
p	$4\bar{o}_4$							-1	1	$+1$		

The co-cycle chain with support 124 has a coefficient of -2 corresponding to the element 1. This is reflected in the row overlap of two stamps which do not intersect at a diagonal entry.

To illustrate further, the array A is constructed for the cycle chain $x = (1, 0, 1, 2)$ and the co-cycle chain $y = (-2, 1, 0, 1)$, showing non-intersecting stamps with both column and row overlaps:

		p^* $2\bar{o}_1 o_2$		s^* $\bar{1}o_1$		s^* $3\bar{o}_2$		s^* $\bar{1}o_4$		s^* $\bar{3}o_3$		p^* $4\bar{o}_3\bar{o}_4$	
p	$2\bar{o}_1$	$+1$	$+1$	-1									
s	$1o_1o_4$		-1	1	-1			-1	$+1$			-1	
p	$4\bar{o}_4$								-1			1	$+1$
p	$\bar{2}o_1$	-1	$+1$			-1							
s	$3\bar{o}_2o_3$		-1			1	$+1$			-1	$+1$	-1	
p	$4\bar{o}_3$										-1	1	$+1$

The four complete non-intersecting stamps are seen clearly in this array.

5 Orientation

If $\bar{B}$ is a regular di-bondgraph the circuits of $M(\bar{B})$ are in one to one correspondence with the primitive chains of $N_{cy}(\bar{B})$ and the co-circuits of $M(\bar{B})$ are in one to one correspondence with the primitive chains in $N_{co}(\bar{B})$. In each row of the circuit matrix of $M(\bar{B})$ add negative signs according to the non-zero coefficients of the primitive cycle chain corresponding to that circuit. Dually for each row of the co-circuit matrix add negative signs according to the non-zero coefficients of the corresponding primitive co-cycle chain. The rows of the signed circuit and co-circuit matrices are respectively called the *oriented cycles* and *oriented co-cycles* of the regular di-bondgraph $\bar{B}$. These are denoted in the abbreviated method used for primitive chains.

Theorem 6 Let $\bar{B}$ be a simple proper regular di-bondgraph. The following are equivalent:

- $\bar{B}$ is orthogonal.
- $\bar{B}$ induces an orientation of the cycle matroid of the underlying bondgraph, via the chains of $N_{cy}(\bar{B})$.
- The oriented cycles are orthogonal to the oriented co-cycles.

Proof: The cycle chains and co-cycle chains are orthogonal if and only if $\bar{B}$ is orthogonal, by Theorem 5, and this is equivalent to the orthogonality of the oriented cycles with oriented co-cycles, which are the rows of the signed circuit and co-circuit matrices. Hence the di-bondgraph induces an orientation of the cycle matroid of the underlying bondgraph.

The orientation of the cycle matroid of the underlying bondgraph of an orthogonal regular di-bondgraph is called the *induced orientation.*

Example 7 Consider the di-bondgraph $\bar{B}$ of Figure 9. The oriented cycles:

$$\{123, 3456, 56\bar{7}, 7\bar{8}\}$$

and oriented co-cycles:

$$\{1\bar{2}, 1\bar{3}4, \bar{4}578, \bar{4}678\}$$

generate the cycle and co-cycle chain groups of $\bar{B}$. The rows of the signed circuit matrix of $\bar{B}$:

1	2	3	4	5	6	7	8
1	1	1	0	0	0	0	0
0	0	1	1	1	1	0	0
0	0	0	0	1	1	−1	0
0	0	0	0	0	0	1	−1

are orthogonal to the rows of the signed co-circuit matrix:

1	2	3	4	5	6	7	8
1	−1	0	0	0	0	0	0
1	0	−1	1	0	0	0	0
0	0	0	−1	1	0	1	1
0	0	0	−1	0	1	1	1

The di-bondgraph induces an orientation of the cycle matroid of the underlying bondgraph.

A di-bondgraph which contains one or more inwardly directed external bonds is called a *pseudo-orthogonal* di-bondgraph. A pseudo-orthogonal di-bondgraph has the same capabilities as an orthogonal di-bondgraph formed from it by reversing the inward external bonds. It is customary in physical system models to use inwardly-directed external bonds when representing source components or for one of the bonds on a 2-port element. These choices are supposed to be connected to the apparent power flows in these bonds, however such superficial reasoning has no mathematical or physical significance. The real benefit in using such inwardly-directed bonds is for convenience when writing the component equations. For instance for 2-ports the equations are symmetric rather than skew-symmetric. It can be shown that a true orientation of the topological structure of a physical system is not required to represent polarities of the component variables. Separate (independent) integral representations of this structure and the dual structure are sufficient. It is not necessary that these representations be orthogonal, and, in the di-bondgraph context, it is quite natural to use pseudo-orthogonal representations.

6 Di-Bondgraphs and Directed Graphs

Theorem 7 Let $\bar{B}$ be a regular di-bondgraph such that the underlying bondgraph B is graphic and let G be an associated graph for B [42]. By directing the edges of G it is possible to construct a di-graph $\bar{G}_{cy}$ whose oriented circuit matrix is the same as the signed circuit

matrix of $\bar{B}$. Dually there is another directed version of G, $\bar{G}_{co}$, whose oriented cutset matrix is the same as the signed co-circuit matrix of $\bar{B}$. The directions on the edges of $\bar{G}_{co}$ may be obtained from those of $\bar{G}_{cy}$ by reversing the directions on any edges corresponding to external bonds of $\bar{B}$ which are directed into the junction. $\bar{B}$ is orthogonal if and only if these two associated di-graphs can be chosen the same.

Proof: The di-graph $\bar{G}_{cy}$ can be formed from G in the usual way, using the signed circuit matrix of $M(\bar{B})$. Similarly $\bar{G}_{co}$ can be formed from G using the signed co-circuit matrix of $M(\bar{B})$. $\bar{G}_{cy}$ and $\bar{G}_{co}$ will be the same if and only if the oriented circuits and cutsets are orthogonal, which is equivalent to $\bar{B}$ being orthogonal. Reversing the direction of an external bond on an s-junction of $\bar{B}$ correponds to changing the sign of the coefficient of the corresponding entry in the signed circuit matrix, which is equivalent to changing the direction of the corresponding edge in $\bar{G}_{cy}$. Dually, if an external bond is on a p-junction, reversing its direction changes the sign of each entry in the signed co-circuit matrix, which is equivalent to changing the direction of an edge in $\bar{G}_{co}$.

Corollary 1 Let $\bar{B}$ be a pseudo-orthogonal di-bondgraph. If, for each bond directed into a p-junction the corresponding entry in each dual chain is multiplied by -1, and, for each bond directed into an s-junction the corresponding entry in each chain is multiplied by -1, the modified chains and dual chains will be orthogonal. This orthogonal structure is also obtained by reversing the directions of all inwardly directed bonds of $\bar{B}$.

A di-graph $\bar{G}_{cy}$ whose oriented circuit matrix is the same as the signed circuit matrix of a regular graphic di-bondgraph $\bar{B}$ is called a *cycle di-graph associated with* $\bar{B}$. Dually a di-graph $\bar{G}_{co}$ whose oriented cutset matrix is the same as the signed co-circuit matrix of $\bar{B}$ is called a *co-cycle di-graph associated with* $\bar{B}$.

Example 8 Consider the di-bondgraph $\bar{B}$ of Figure 10. The internal bonds are labelled so that the generating chains of the cycle and co-cycle chain groups may be given explicitly as linear combinations of junction chains and dual junction chains. Generating cycle chains are:

$$\begin{aligned}
\bar{1}23 &= \bar{1}o_1o_6 + \bar{o_1}3 + \bar{o_6}2 \\
34\bar{6} &= o_2o_3\bar{o_7} + \bar{o_2}3 + \bar{o_3}4 + \bar{6}o_7 \\
5\bar{6}7 &= 7\bar{o_9}o_{10} + \bar{6}o_9 + 5\bar{o_{10}} \\
2\bar{4}5 &= o_5\bar{o_4}o_8 + 2\bar{o_5} + \bar{4}o_4 + 5\bar{o_8}
\end{aligned}$$

and generating co-cycle chains are:

$$\begin{aligned}
\bar{2}34 &= 3\bar{o_1}o_2 + \bar{o_2}o_3 + 4\bar{o_3}o_4 + \bar{o_4}o_5 + \bar{2}\bar{o_5}o_6 + \bar{o_6}o_1 \\
\bar{1}3457 &= 4\bar{o_3}o_4 + o_3\bar{o_2} + 3o_2\bar{o_1} + \bar{1}o_1 + \bar{o_4}o_8 + 5\bar{o_8}o_{10} + 7\bar{o_{10}} \\
1\bar{2}46\bar{7} &= 4\bar{o_3}o_4 + o_3\bar{o_7} + 6o_7\bar{o_9} + \bar{7}o_9 + \bar{o_4}o_5 + \bar{2}\bar{o_5}o_6 + 1\bar{o_6}
\end{aligned}$$

The rows of the signed circuit matrix:

1	2	3	4	5	6	7
−1	1	1	0	0	0	0
0	0	1	1	0	−1	0
0	0	0	0	1	−1	1
0	1	0	−1	1	0	0

are the oriented cycles of $M(\bar{B})$ and the rows of the signed co-circuit matrix:

1	2	3	4	5	6	7
0	−1	1	1	0	0	0
−1	0	1	1	1	0	1
1	−1	0	1	0	1	−1

are the oriented co-cycles of $M(\bar{B})$. Associated cycle and co-cycle di-graphs are also shown. The directions on the edges of $\bar{G}_{co}$ may be obtained from those of $\bar{G}_{cy}$ by reversing the directions on edges 1, 4, 5 and 6, corresponding to the external bonds of $\bar{B}$ which are directed into the junction.

7 Summary

A mathematical basis has been presented for the use of di-bondgraphs as diagrams of combinatorial information for physical system models. A di-bondgraph is a pictorial device which defines integral representations of the cycle and co-cycle matroids that represent the component interconnections of a physical system. Such integral representations are required to define system variable polarities, so that continuity equations (generalized Kirchhoff laws) may be expressed to represent spatial constraints between the variables. In contrast, a non-directed bondgraph represents only the interconnection topology of the system components themselves by defining binary representations of the cycle and co-cycle matroids.

There is no *a priori* connection between a bondgraph B and a directed version $\bar{B}$, the same bondgraph with half-arrows added to the bonds. In general, the matroids of B and $\bar{B}$ are unrelated, i.e. they represent independent combinatorial information. In particular, this implies that the addition of half-arrows to the bonds of B can have no general significance with respect to orienting the combinatorial structure represented by B. However, an important special class can be defined, the regular di-bondgraphs, for which B and $\bar{B}$ always give representations of the same matroid. Such di-bondgraphs are orientable and the half-arrows induce an orientation on the cycle and co-cycle matroids in the precise combinatorial sense [37]. If the cycle matroid, $M(B)$, is graphic this orientation may be used to assign directions to the edges of an associated cycle graph for B, giving an associated cycle di-graph for $\bar{B}$. Dually, a graphic co-cycle matroid can be associated with a co-cycle di-graph for $\bar{B}$.

A di-bondgraph $\bar{B}$ is orthogonal if every external bond is directed outward and otherwise pseudo-orthogonal. A regular orthogonal di-bondgraph has orthogonal oriented cycles and

co-cycles. In the case of a regular graphic pseudo-orthogonal di-bondgraph with associated graph G, it is possible to direct the edges of G and construct a di-graph $\bar{G}_{cy}$, called an oriented cycle di-graph, so that the signed circuit matrix of $\bar{G}_{cy}$ and cycle matrix of $\bar{B}$ are identical. Dually, a di-graph $\bar{G}_{co}$, called an oriented co-cycle di-graph, can be constructed so that the signed cutset matrix of $\bar{G}_{cy}$ and co-cycle matrix of $\bar{B}$ are identical. Oppositely-directed edges in a pair of signed cycle and co-cycle di-graphs correspond to those bonds which are directed inward in the di-bondgraph $\bar{B}$. A di-bondgraph is orthogonal if and only if the signed cycle and co-cycle di-graphs can be chosen to be identical. (Pseudo-)orthogonality for di-bondgraphs is the crucial property for their use in modelling physical systems, as this ensures that the product of respective dual variable pairs is a conserved quantity.

A di-bondgraph is non-regular when its cycle matroid is a non-regular matroid, for instance the uniform matroid of rank 2 on four elements, represented by $\bar{B}_{2,4}$. A system whose structure is represented by a non-regular di-bondgraph cannot be modelled explicitly using a linear graph to define the topological structure, because the cycle matroid may not be oriented and has no associated graph (or di-graph). This clarifies the statements in [58] and [41], in which the lack of an orientation is cited as sufficient for implying a non-physical di-bondgraph. In fact, a physical system model requires only a pair of integral representations of a matroid and its dual – whether or not these actually provide an orientation is irrelevant. This situation is confirmed by the occasional use of non-regular di-bondgraphs in physical applications, for instance in models of underspecified systems such as the differential gearbox shown in Figure 2 of [59], which uses the non-regular di-bondgraph $\bar{B}_{2,4}$ (see our Fig.8). Non-regular di-bondgraphs have no combinatorial connection with their underlying bondgraphs, reflecting the fact that no binary representation is possible for their matroids. However, as for all di-bondgraphs, non-regular di-bondgraphs continue to provide integral representations of the pair of dual matroids, and cycles and co-cycles are always orthogonal to each other when $\bar{B}$ is orthogonal.

Any di-bondgraph, regular or non-regular, may potentially be used to represent the topological structure of a physical system [39], however almost all bondgraph system modelling applications in practice involve regular di-bondgraphs. Therefore it can be concluded that bondgraph and linear graph methodologies have identical capabilities as combinatorial models of physical systems.

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A Integral Chain Groups and Matroids

Chain Group Matroids

Let S be a finite set and F either a field or the ring of integers. A *chain on S over F* is a map $f : S \to F$. The set of all chains on S over F will be denoted by $L(S, F)$, or $L(S)$ if F is clear from the context. An *integral chain* is a chain over the ring of integers and a *binary chain* is a chain over the field $GF(2)$ (which consists of elements 0 and 1 and standard arithmetical operations on these). The *support* of the chain f is

$$\mathrm{supp}(f) = \{x \in S \mid f(x) \neq 0\}.$$

The *zero chain*, denoted by 0, maps every element of S to 0. The *sum*, $f + g$, of two chains f and g and the *scalar product*, $c \cdot f$, of $c \in F$ with $f \in L(S)$ are defined by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (c \cdot f)(x) &= c(f(x)). \end{aligned}$$

A set, N, of chains in $L(S,F)$, is called a *chain group on S over F* if N is closed under sums and scalar multiplication. An *integral chain group* is a chain group of integral chains and a *binary chain group* is a chain group of binary chains.

A non-zero chain, f, in a chain group N, is an *elementary chain* if there is no chain $g \in N$ so that $\mathrm{supp}(g)$ is properly contained in $\mathrm{supp}(f)$.

It can be shown that the supports of chains in a chain group, N, are the dependent sets of a matroid, called the *matroid of the chain group* and denoted by $M(N)$. The circuits of $M(N)$ are the supports of the elementary chains of N.

Let f and g be chains in $L(S)$. The *inner product* of f with g is defined by

$$\langle f \mid g \rangle = \sum_{x \in S} f(x)g(x).$$

The chains f and g are *orthogonal* if $\langle f \mid g \rangle = 0$.

Binary and Integral Matroids

A *binary matroid* is a matroid which is isomorphic the matroid of a binary chain group. An *integral matroid* is a matroid which is isomorphic the matroid of an integral chain group.

If A is matrix with entries in F, a field or the ring of integers, and the matroid M is isomorphic to the matroid induced on the columns of A by linear independence then M is *representable over F* and A is a *matrix representation* of M. If F is $GF(2)$ or the integers then A is a *binary representation* or *integral representation* respectively.

Theorem M-1 A matroid M is representable over a field F if and only if M is isomorphic to the matroid $M(N)$ of a chain group N over F.

Theorem M-2 A matroid M is a binary matroid if and only if it is representable over $GF(2)$.

The *Fano matroid, M(Fano)*, a binary matroid, is the matroid of the chain group generated by the chains

1	2	3	4	5	6	7
1	0	0	0	1	1	0
0	1	0	0	0	1	1
0	0	1	0	1	0	1
0	0	0	1	1	1	1

Equivalently *M(Fano)* is the matroid with minimal generating set $\{1237, 136, 124, 235\}$.

Theorem M-3 The Fano matroid, *M(Fano)*, and its dual are representable only over $GF(2)$.

Theorem M-4 The Fano matroid and its dual are the smallest binary matroids which are neither graphic nor co-graphic.

Let S be a set with $\sharp S = n$ and define a subset, T, of S to be independent if $\sharp T \leq k$. This defines a matroid, $U_{k,n}$, on S called the *uniform matroid of rank k*. The bases of $U_{k,n}$ are all subsets of S of cardinality k and the circuits of $U_{k,n}$ are all subsets of S of cardinality $k+1$.

Theorem M-5 The uniform matroid, $U_{2,4}$, is representable over every field except $GF(2)$.

Example 9 $U_{2,4}$ is the matroid of the chain group

1	2	3	4
1	0	−1	−1
0	1	−1	1
1	1	−2	0
1	−1	0	−2

which is generated by the first two chains shown. $U_{2,4}$ is self-dual and the first two rows above are an integral representation of its dual.

The *circuit matrix*, $D(M)$, of a binary matroid, M, is the incidence matrix of circuits against elements. Dually the *co-circuit matrix*, $D^*(M)$, is the incidence matrix of co-circuits against elements. M is called *orientable* if it is possible to change some of the non-zero entries of D and D^* to -1 in order to make the rows of D orthogonal to the rows of D^* *over the integers*. The signed circuit and co-circuit matrices are called an *orientation* of M.

A graphic matroid M, isomorphic to the cycle matroid of a graph G say, is orientable. After directing the edges of G it is simple to insert appropriate negative signs in the circuit and co-circuit matrices of M and M^* to make them orthogonal over the integers. There are orientable non-graphic matroids, for instance the co-cycle matroid of K_5, the complete graph on five vertices. There are also non-orientable binary matroids, for instance the Fano matroid.

An orientation of a binary matroid M can be used to obtain integral representations of both M and its dual. So an orientable matroid must be integral. Integral representations, A and A^* of M and its dual respectively, can be used to obtain an orientation of M provided the rows of A are orthogonal to the rows of A^* and these matrices contain only 0 and ± 1 entries.

Regular Matroids

Let N be an integral chain group on S. A *primitive chain* is a chain which takes only the values -1,0 and 1. N is a *regular* chain group if, for every elementary chain, there is a primitive chain with the same support. A matroid is *regular* if it is isomorphic to the matroid of some regular chain group.

The matroid of a non-regular chain group may be a regular matroid, as shown in the following example.

Example 10 Let N be the chain group generated by the chains x_1 and x_2 shown below:

	1	2	3	4	
x_1	2	-1	0	1	124
x_2	1	0	1	0	13
$2x_2 - x_1$	0	1	2	-1	234
$x_1 - x_2$	1	-1	1	-1	1234

This chain group is non-regular since, for instance, there is no primitive chain with the same support as the elementary chain $(2, -1, 0, 1)$. $M(N)$, the matroid of N, has circuits $\{124, 13, 234\}$ corresponding to the supports of elementary chains of N. However M is a regular matroid since

	1	2	3	4	
y_1	1	1	0	1	124
y_2	1	0	1	0	13
$y_1 - y_2$	0	1	-1	1	234

defines a regular chain group of which M is the matroid.

Theorem M-6 A matroid is regular if and only if it is representable over every field. In particular if a matroid is regular then it is binary.

Theorem M-7 A matroid is regular if and only if it is orientable.

Let M be a matroid on the set S and $T \subseteq S$ with complement $T' = S - T$. The *restriction* of M to T, denoted by $M \mid T$, is the matroid whose circuits are all circuits of M which are contained in T. The restriction $M \mid T$ is said to be obtained from M by *deleting* T' and is also denoted by $M \times T'$. The *contraction* of M to T, denoted by $M \cdot T$, is the matroid whose circuits are the minimal non-null sets of the form $C \cap T'$, where C is a circuit of M. The contraction $M \cdot T$ is said to be obtained from M by *contracting* T' and is also denoted by $M \circ T'$. A matroid is a *minor* of M if it obtained from M by any combination of restrictions and contractions of subsets of S.

Theorem M-8 (Tutte [60]) A binary matroid is regular if and only if it does not contain as a minor either the Fano matroid or its dual.

Theorem M-9 A matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$.

Theorem M-10 A matroid is regular if and only if it is integral and binary.

Proof: If M is integral it is isomorphic to the matroid of an integral chain group N. In particular every minor of M must also be integral. If such an M were non-regular then, by Theorem M-8, it would have a minor isomorphic to either *M(Fano)* or its dual, neither of which are integral matroids. Thus an integral binary matroid must be regular. The converse follows from Theorem M-6.

Figure Captions

Figure 1. Di-bondgraph which provides an orientation for the underlying bondgraph.

Figure 2. Di-bondgraph which does not provide an orientation for the underlying bondgraph.

Figure 3. Di-bondgraph with regular cycle matroid and non-regular cycle chain group.

Figure 4. Classes of matroids relevant to bondgraphs and di-bondgraphs. The binary matroids correspond to bondgraphs and the integral matroids correspond to di-bondgraphs. Regular matroids are binary and integral, and may be graphic, co-graphic, or both.

Figure 5. Contracted pair of adjacent degnerate internal junctions.

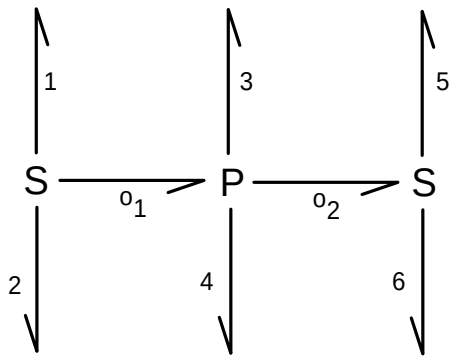
Figure 6. Di-bondgraph segment in proof of Theorem 5.

Figure 7. Regular orthogonal di-bondgraph to illustrate proof of Theorem 5.

Figure 8. Non-regular orthogonal di-bondgraph used in illustration of the proof of Theorem 5.

Figure 9. Orthogonal di-bondgraph.

Figure 10. Di-bondgraph and associated cycle and co-cycle di-graphs.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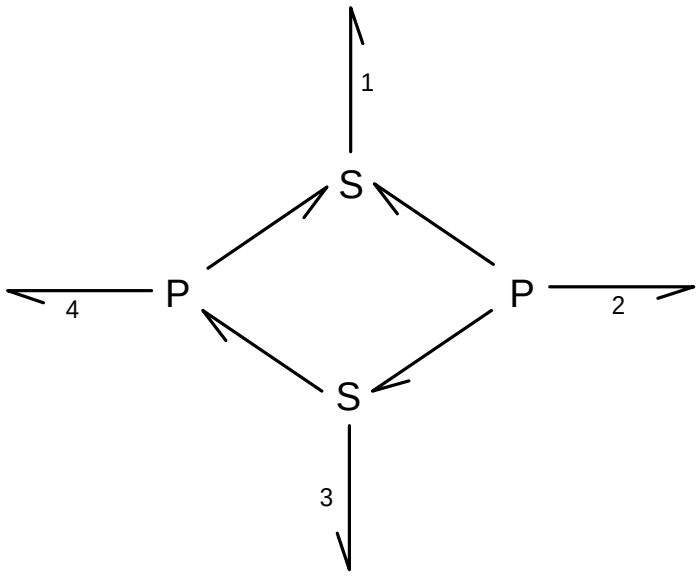
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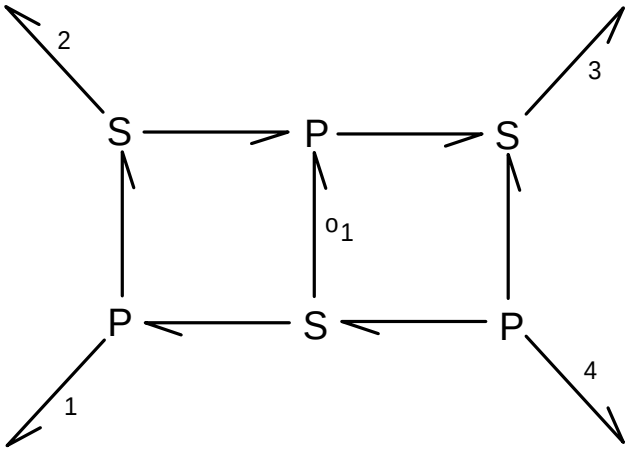
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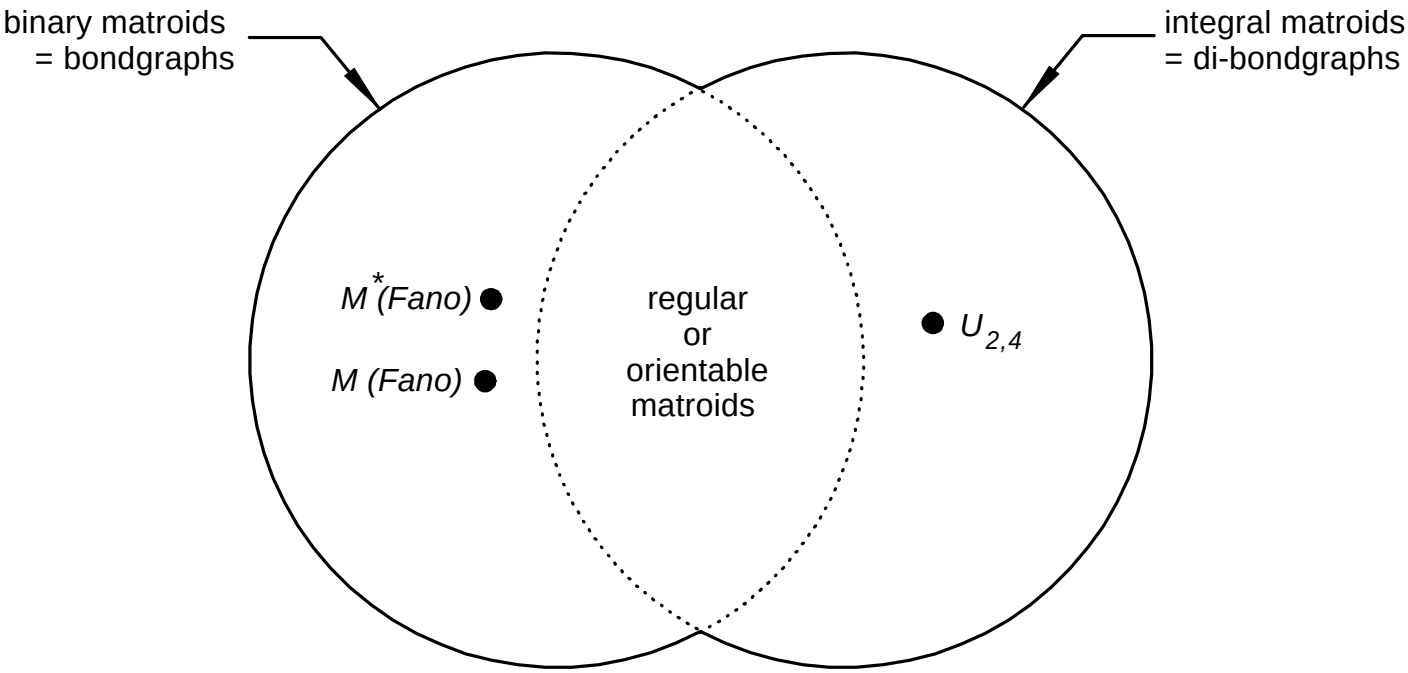
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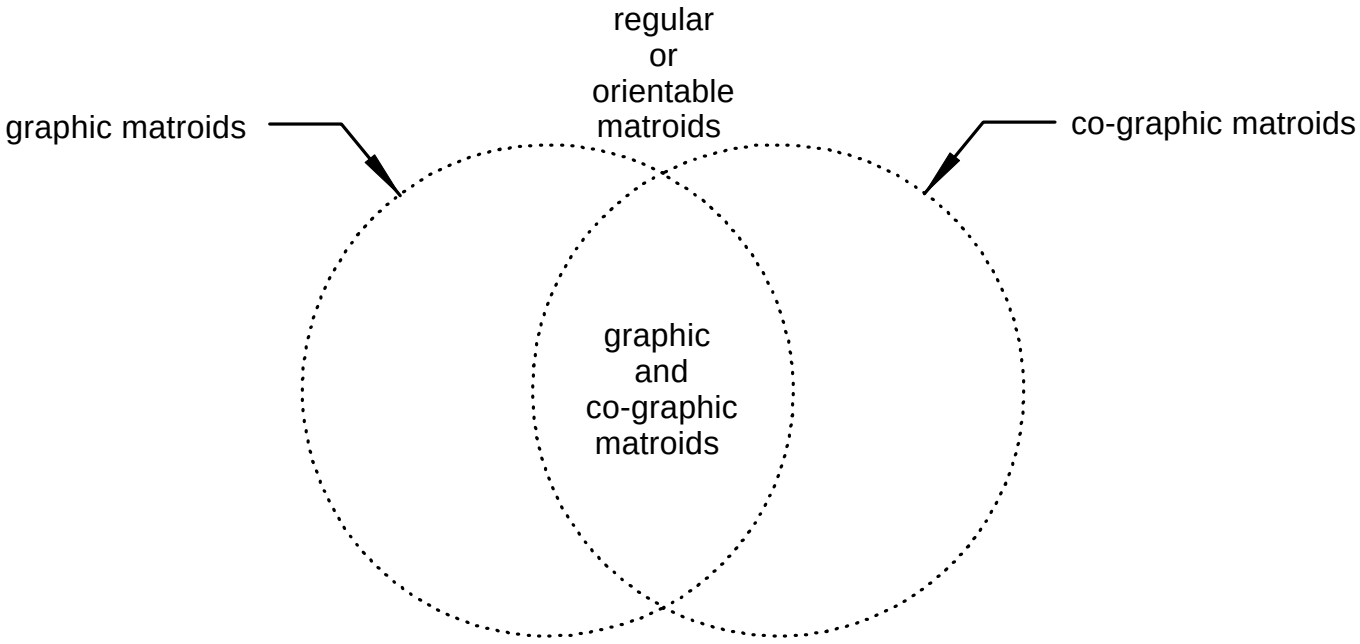
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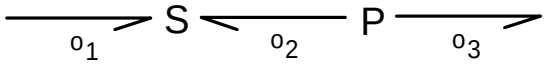
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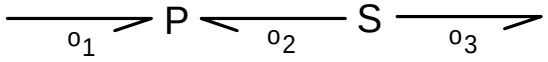
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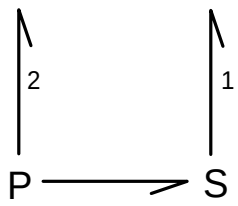
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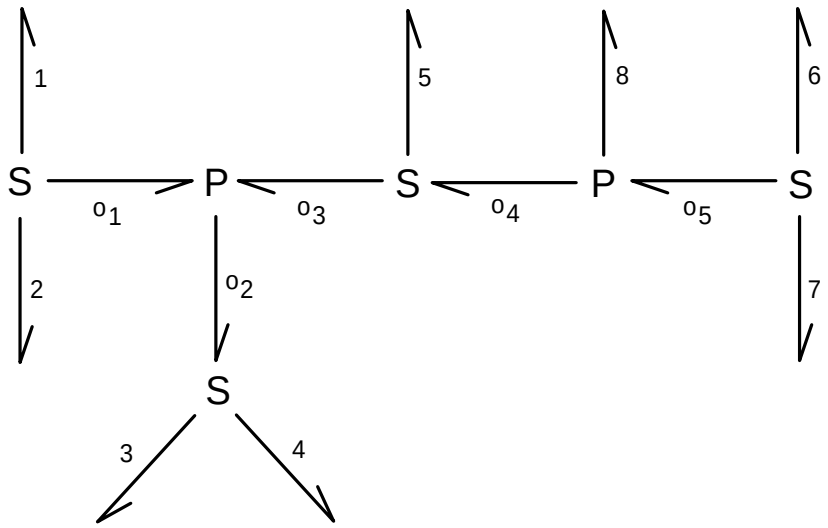
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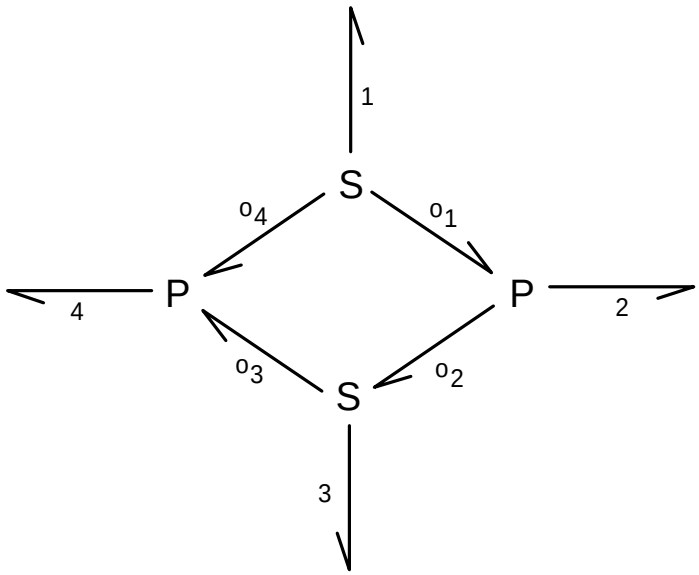
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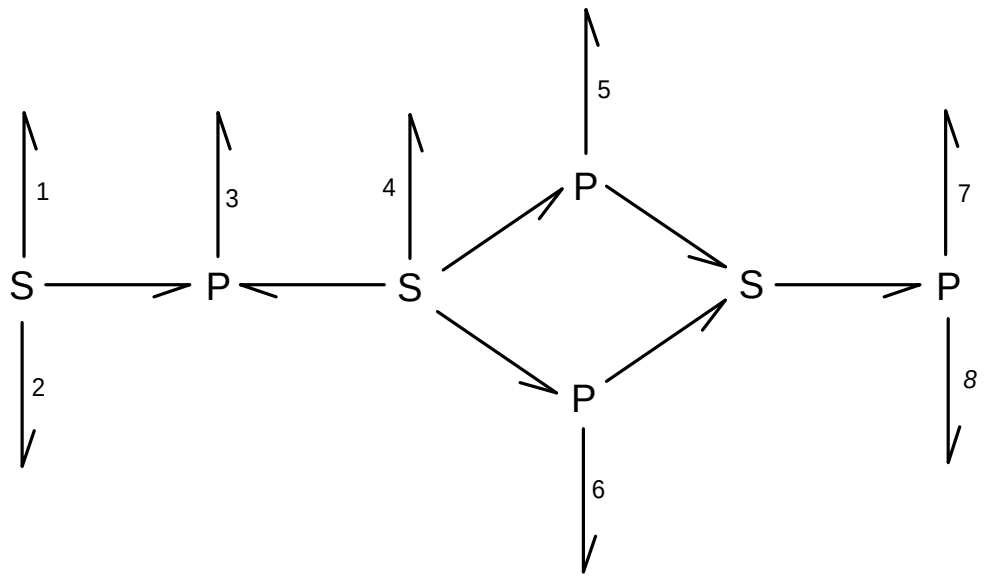
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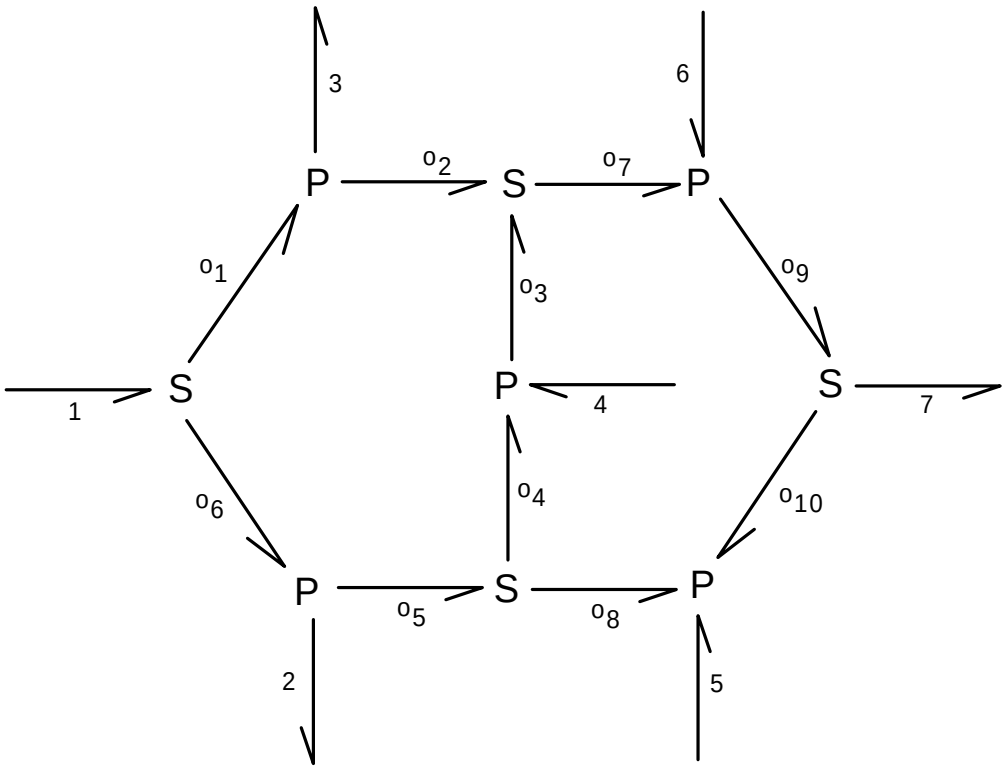
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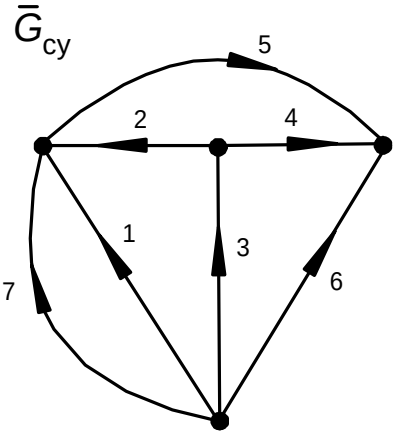
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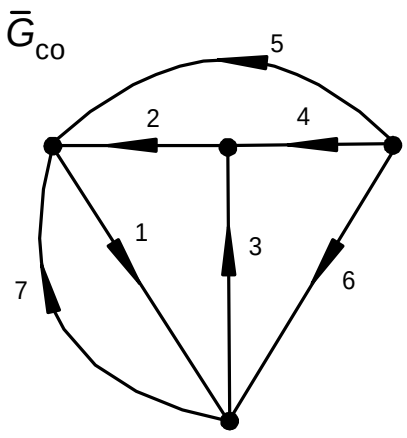
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