

# *The Mathematical Foundations of Bond Graphs—IV. Matrix Representations and Causality*

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**ABSTRACT:** *The development of a mathematical theory for bond graphs continues with an analysis of two areas crucial to the derivation of system equations from a bond graph model. Matrix representations of bond graph matroids are examined and used to provide a rigorous proof of the mathematical equivalence of the linear graph and bond graph modelling methods. The procedure of selecting causality by the method of causal strokes is discussed. This is shown to be a device for choosing a base of the cycle matroid of a bond graph, and a sufficient condition is proved for when the method gives a base. An example is given of a bond graph with a causal loop which corresponds to a valid causal assignment. Various combinatorial formulae are proved concerning ranks and dimensions. Formulae for the ranks of effort and flow matrices and the number of independent equations obtained from a bond graph are consequences of the results.*

## **I. Introduction**

In Parts I and II (1, 2), we used elementary mathematical concepts from linear algebra to develop an independent theory for non-directed bond graphs, which may be called *combinatorial bond graphs* since they contain only structural information and no physical information. In forming a system model, physical components are associated with the bonds and the resulting model may be called a *system bond graph*. This is parallel to the derivation of a *system graph* model from a linear graph which provides the structural information.

In Part III (3), the theory was extended and connection between bond graphs and binary matroids established. We defined the matroids of a bond graph, the *cycle matroid* and its dual, the *co-cycle matroid*, and showed how the bond graph is a direct pictorial diagram of this structural matroid information. In Part V (4), the final paper of the series, we will analyse the technique of directing bonds by adding half-arrows, which is crucial to the application of bond graphs to physical system modelling. Under certain conditions this produces an oriented structure, essential for writing topological equations for a system model.

In this paper, we develop two aspects of the theory which are important in physical system modelling. First we demonstrate how, by using matrix representations of bond graph matroids, it is possible to derive the customary structural matrices

which appear in the bond graph literature, the *junction structure matrices*. It is through the use of matrix representations that topological system equations are written. In this context the results of several authors, e.g. (5, 6), are seen to be consequences of theorems we prove for bond graph matroids.

The second part of this paper consists of an analysis of the bond graph technique known in the literature as *selection of causality*, a crucial aspect of bond graph modelling. We show that the method is a graphical means for indicating a combinatorially fundamental collection of bonds for a bond graph. In terms of matroids, this is a *base* of the cycle matroid of the bond graph. To obtain a causal bond graph, causal strokes are added to the bonds according to well-known rules. We prove that the absence of a causal loop is a sufficient condition for producing a base or, in customary bond graph terminology, producing a valid assignment of causality. We also show that this condition is not necessary, by providing a counter-example of a bond graph with a causal loop which gives a valid assignment of causality. This is contrary to the customarily accepted fact that causal loops always lead to invalid causality, see e.g. (7).

The terminology, notation and special symbols defined in the first three parts will be used freely here. For definitions and examples refer to (1–3).

The notation  $\#A$  is used for the number of elements of a set  $A$ . We conclude the Introduction by recalling and expanding on the special notation we used for sets in the earlier parts of this series.

### Notation

In (1), we adopted a convenient juxtaposition notation for sets. This was done to simplify writing a set of sets, which frequently arises in this algebraic theory. For example, we use  $\{123, 234, 1, 23, \emptyset\}$  as a simplified notation for  $\{\{123\}, \{234\}, \{1\}, \{23\}, \emptyset\}$ , where  $\emptyset$  means the empty set  $\{\}$ . We adopt the convention that set braces which are explicitly written always refer to the second level of sets, that is, they refer to a set of sets. This removes any possible ambiguity when the items in the set are sets, each of which has only one element. For example,  $\{1, 2, 3\}$  will mean the set of sets  $\{\{1\}, \{2\}, \{3\}\}$ , whereas  $123$  will mean  $\{1, 2, 3\}$ . These two are quite different objects and it is important to be able to distinguish clearly between them.

## II. Matrix Representation of Matroids

An introduction to matroid theory is given in (3), where we outline the basic definitions and properties of a matroid, and provide some important examples. We recall one definition which will be important here.

*Definition.* The *rank* of a matroid is the number of elements in any base or, equivalently, the number of elements in a maximal independent set. The *co-rank* of a matroid is the rank of the dual matroid.

In this section we examine matrix representations of bond graph matroids. These matroids are binary matroids, for which a matrix representation is a matrix with 0 and 1 entries, whose columns define the matroid. The precise definition is given

below. The results stated in this section are standard results in matroid theory and proofs can be found in [8].

*Definition.* A matrix representation of a binary matroid  $M$  is a matrix  $A$  with entries in  $GF(2)$ , the field  $\{0, 1\}$ , such that  $M$  is isomorphic to the matroid induced on the columns of  $A$  by linear independence.

*Definition.* Suppose that  $M$  is binary matroid on the set  $S$ , with rank  $r$  and co-rank  $q$  ( $r+q$  is equal to the number of elements in  $S$ ). Choose a base  $R = \{b_1, b_2, \dots, b_r\}$  of  $M$  and let  $Q = \{c_1, c_2, \dots, c_q\}$  be the co-base corresponding to  $R$ . Then, using elementary linear algebra, we can see that there will be a matrix representation of  $M$  of the form  $A = [I_r | D]$ , where  $I_r$  denotes the  $r \times r$  unit matrix and the columns of  $A$  are arranged in the order corresponding to  $b_1, b_2, \dots, b_r, c_1, c_2, \dots, c_q$ . Such a matrix representation is called the *standard representation of  $M$  with respect to  $R$* .

*Theorem I*

If  $M$  has a standard representation  $A = [I_r | D]$  with respect to the base  $R$ , then, with respect to the co-base,  $Q$ , corresponding to  $R$ , the dual matroid,  $M^*$ , has the standard representation  $[D' | I_q]$ , where the columns are arranged in the same order as those of  $A$ , and  $D'$  is the transpose matrix of  $D$ .

*Definition.* Let  $M$  be a matroid with base  $R$  and corresponding co-base  $Q$ . It can be shown that, for each co-base element  $c$  there exists a unique circuit,  $C = C(c, R)$ , which contains  $c$  and no other co-base element. That is, the circuit  $C$  contains only  $c$  and elements in the base  $R$ . Dually, for each base element  $b$  there exists a unique co-circuit,  $C^* = C^*(b, Q)$ , which contains  $b$  and no other base element. The circuit  $C(c, R)$  is called the *fundamental circuit of  $c$  with respect to  $R$* . The co-circuit  $C^*(b, Q)$  is called the *fundamental co-circuit of  $b$  with respect to  $Q$* .

*Theorem II*

Let  $R = \{b_1, b_2, \dots, b_r\}$  be a base of a binary matroid  $M$  with corresponding co-base  $Q = \{c_1, c_2, \dots, c_q\}$ . Define the matrix  $A$  by  $A = [I_r | D]$ , where  $D_{km} = 1$  if  $b_k$  is in the fundamental circuit of  $c_m$  with respect to  $R$ , and otherwise  $D_{km} = 0$ . Then  $A$ , determined uniquely by  $R$ , is the standard matrix representation of  $M$  with respect to the base  $R$ .

*Example 1.* We illustrate the results and definitions above by considering the cycle matroid,  $M(G)$ , and its dual, the co-cycle matroid,  $M^*(G)$ , of the linear graph  $G$  shown in Fig. 1. Let the base  $R = 123$ .

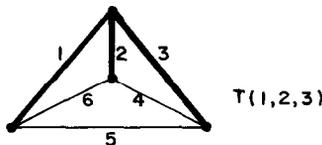


FIG. 1. Linear graph with tree.

Then the corresponding co-base is  $Q = 456$ . The fundamental circuits of  $M(G)$  with respect to  $R$  are  $\{423, 513, 612\}$ . Thus the standard representation of  $M(G)$  with respect to  $R$  is the matrix

$$A(M) = \begin{bmatrix} & 1 & 2 & 3 & & 4 & 5 & 6 \\ \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right. \end{bmatrix}.$$

The standard representation of  $M^*(G)$  with respect to  $Q$  is the matrix

$$A(M^*) = \begin{bmatrix} & 1 & 2 & 3 & & 4 & 5 & 6 \\ \left[ \begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right. \end{bmatrix}.$$

The rows of  $A(M)$  correspond to the fundamental co-cycles of the graph  $G$  and the rows of  $A(M^*)$  correspond to the fundamental cycles of the graph  $G$ . Thus, in the terminology of graph theory, these matrices are fundamental cutset and circuit matrices for the graph  $G$ , with respect to the spanning tree  $T(1, 2, 3)$ . This spanning tree, of course, is the base 123 of the cycle matroid of  $G$ .

We can interpret the standard matrix representation,  $A(M)$ , of a matroid  $M$  as follows: the  $k$ th row of  $A(M)$  corresponds to the  $k$ th fundamental co-circuit of the matroid  $M$ . Dually, the  $k$ th row of  $A(M^*)$  corresponds to the  $k$ th fundamental circuit of  $M$ .

*Remark 1.* Theorem II is true for representations of matroids over fields other than  $GF(2)$ , but, in general,  $D'$  is replaced by  $-D'$  in the statement of the theorem. In the case of  $GF(2)$ ,  $D' = -D'$  since  $1 = -1$  in this field. We discuss this further when we consider directed bond graphs in Part V (4).

### III. Matrix Representations of Bond Graph Matroids

The cycle and co-cycle matroids of a bond graph are dual binary matroids. Thus we can apply the result of Theorem II to these matroids, and obtain matrix representations of them.

#### Theorem III

Let  $B$  be a bond graph and  $G$  any graph associated with  $B$ . Then  $M(B)$ , the cycle matroid of  $B$ , and  $M(G)$ , the cycle matroid of  $G$ , have identical standard representations with respect to a given base for each matroid, when the bases are ordered corresponding to the isomorphism between  $M(B)$  and  $M(G)$ .

*Proof:* The two cycle matroids may be regarded as the same matroid, by numbering the edges and bonds in such a way that there is an exact correspondence via the isomorphism. Provided the numbering is suitable, then, the representations are identical matrices.

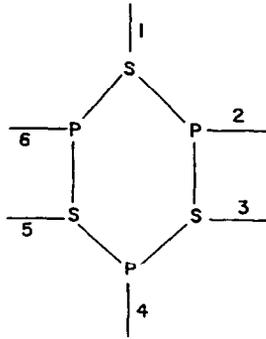


FIG. 2. Bond graph for Example 2.

*Remark 2.* In linear graph or bond graph modelling the interconnection topology of the system is modelled by a matroid structure, the cycle (and co-cycle) matroid. For formulation methods, a matrix representation of this matroid structure is used to write the structural or topological system equations. The fact that these matrices are identical for the two models is reflected in the exact parallel between the two modelling techniques, when the same formulation method is applied to a bond graph or equivalent linear graph model of a physical system. Thus *Theorem III* is a theoretical explanation for the equivalence of the two modelling methods. Of course, the representation need not be explicitly written in matrix form, but, however the structural equations are written, it is the matrix representation which is being used.

*Example 2.* The bond graph,  $B$ , shown in Fig. 2 is associated with the linear graph used in Example 1. The numbering of the bonds is in exact correspondence and so  $M(B) = M(G)$ . The standard matrix representation of  $M(B)$  with respect to the base 123 is the matrix  $A(M)$  given in Example 1. Dually,  $A(M^*)$  is the standard matrix representation of  $M(B^*)$  with respect to the co-base 456.

*Remark 3.* In the bond graph literature *junction structure matrices* are used to describe the topological relationship between the flow and effort variables of a bond graph model. These are the *effort* and *flow* matrices defined, for instance, in (5). The matrix representations described above may be derived from these junction structure matrices when the variables corresponding to internal bonds are eliminated. After the discussion of causality in the next section, we explain the properties of these matrices in the context of our results on bond graph matroids.

#### IV. Base Selection for Bond Graph Matroids

The selection of a base for the cycle matroid of a linear graph is accomplished by choosing a spanning tree (or forest if the graph is non-connected) for the graph. This is a well-known procedure and needs no explanation. If the linear graph is used in a model for a physical system as a system graph, the tree is called a *formulation tree*.

For bond graphs the situation is quite different. In physical system modelling using system bond graphs the selection of a base is crucial to the standard state equation formulation method. We establish here that the technique known in the bond graph literature as *selection of causality* corresponds precisely to choosing a base for the cycle matroid of a bond graph. Furthermore, we provide a theoretical justification for this procedure by giving a sufficient condition for the method to produce a valid base. In spite of the importance of this technique to bond graph modelling, such a justification for selection of causality has not appeared in the literature.

Let  $B$  be a simple bond graph. Without loss of generality we assume that  $B$  is proper, for otherwise we simply consider the proper contraction of  $B$ , which is an equivalent proper bond graph. The proper contraction of a bond graph has precisely the same cycle and co-cycle matroids as the original bond graph.

In (2), we showed how to construct the *quotient space* of a vector space with a subspace. In particular, we defined the quotient space  $W/W_s$ , where  $W$  is the bond space of  $B$  and  $W_s$  is the  $s$ -space of  $B$ .

*Notation.* Following the notation of (1), we use  $n(B)$  to denote the total number of all bonds of  $B$ , and  $e(B)$  and  $i(B)$  to denote respectively the numbers of external and internal bonds of  $B$ . It is convenient to use the subscripts  $s$  and  $p$  to modify these as follows:

$e_s(B)$  = the number of external bonds incident on an  $s$ -junction of  $B$ ,  
 $e_p(B)$  = the number of external bonds incident on a  $p$ -junction of  $B$ .

We usually omit the specific reference ( $B$ ) in the notation. Note that we have

$$n = e + i$$

and

$$e = e_s + e_p.$$

We also recall that  $p(B)$  and  $s(B)$  denote respectively the numbers of  $p$ - and  $s$ -junctions of  $B$ .

*Theorem IV*

Let  $B$  be a simple proper bond graph. The dimension of the quotient space  $W/W_s$  is given by

$$\dim W/W_s = p - s + e_s.$$

*Proof:* We first calculate the dimension of  $W_s$ . This is equal to the number of independent  $s$ -type and  $p$ -type vectors of  $B$ , since the collection of these elementary junction vectors is a basis for  $W_s$ . The  $s$   $s$ -type vectors are all independent but not all of the  $p$ -type vectors are independent. In fact, for each  $p$ -junction the number of independent  $p$ -type vectors obtained is equal to the number of bonds incident on that junction less one. Hence, considering all the  $p$ -junctions of  $B$ , we determine that the number of independent  $p$ -type vectors is the total number of all bonds on  $p$ -junctions less the number of  $p$ -junctions of  $B$ . (Note that we have used the fact that  $B$  is proper.)

Now, since  $B$  is proper and simple, each internal bond of  $B$  corresponds to precisely one  $p$ -junction of  $B$ . This follows because no internal bond is incident on two  $p$ -junctions and there are no parallel internal bonds. Also, since  $B$  is simple, there is no single isolated  $p$ -junction, which would have no incident internal bond. Thus we conclude that there are precisely  $i + e_p$  bonds incident on all the  $p$ -junctions of  $B$ .

Combining the results above, we obtain a formula for the dimension of  $W_s$  :

$$\begin{aligned} \dim W_s &= s - p + i + e_p \\ &= s - p + i + e - e_s \\ &= s - p + n - e_s. \end{aligned}$$

It is a standard result in linear algebra that

$$\dim W/W_s = \dim W - \dim W_s.$$

Now  $\dim W$  is  $n$ , hence we have

$$\begin{aligned} \dim W/W_s &= n - \dim W_s \\ &= p - s + e_s, \end{aligned}$$

using the result above. This is the formula given in the theorem.

*Theorem IV\**

The dual of Theorem IV gives

$$\dim W/W_p = s - p + e_p.$$

The next theorem is an important algebraic result which we need. See (9) for a proof. The symbol  $\approx$  is used for isomorphism of vector spaces.

*Theorem V*

Let  $X$  and  $Y$  be subspaces of a vector space  $W$ . Then the following are equivalent :

- (i)  $W = X \oplus Y$ ,
- (ii)  $Y \approx W/X$ ,
- (iii)  $Y$  contains exactly one element from each coset of  $X$ .

Theorem V will be applied to the bond space,  $W$ , of a bond graph, by considering the quotient space of  $W$  with the  $s$ -space  $W_s$ . We shall construct the subspace  $Y$ , complementary to  $W_s$ , by considering the procedure of causality selection for a bond graph.

Rather than using the usual causal stroke notation, in this section we use a different and more convenient notation for causal selection on a bond graph. In the next section we explain the connection between these and discuss some results of other authors, in the context of our results.

*Definition.* Let  $B$  be a bond graph. Suppose that it is possible, for each junction of  $B$ , to choose a distinguished bond incident on that junction, in such a way that each junction has *precisely* one distinguished bond incident on it. The distinguished

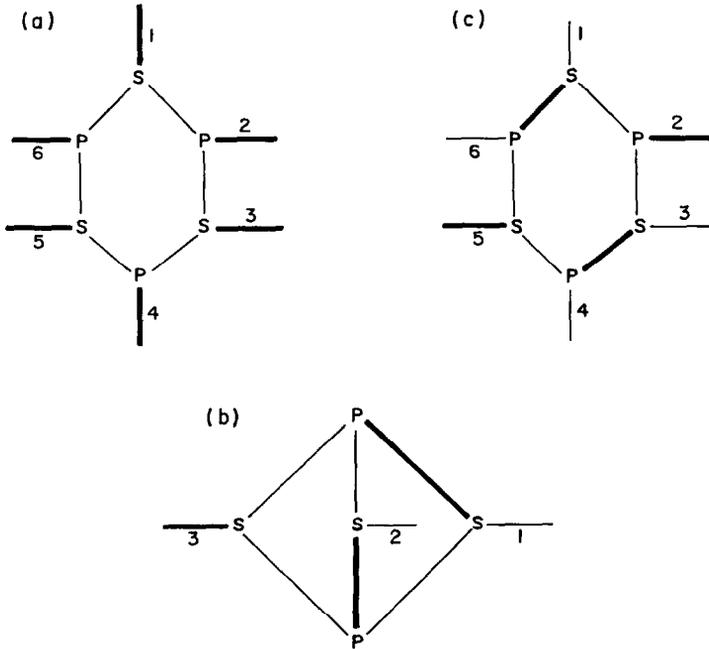


FIG. 3. Pseudo-base colourings of bond graphs.

bond will be indicated in bond graph diagrams by a heavy line and we refer to this bond as being *coloured red*. All the other bonds will be said to be *coloured white*. A colouring of the bonds of  $B$  with this property will be called a *pseudo-base colouring of  $B$* .

*Theorem VI*

A pseudo-base colouring exists for any simple bond graph.

*Proof:* We construct a pseudo-base colouring for  $B$ . Suppose first that  $B$  has no internal junctions. Then, by choosing precisely one external bond on each junction and colouring it red, we obtain a pseudo-base colouring of  $B$ .

Now suppose that  $B$  has internal junctions. We first colour  $B$  according to the method above, with one red external bond on each non-internal junction. Then we introduce a red bond onto each internal junction as follows. For each adjacent pair of internal junctions we colour red the common incident (internal) bond. Any remaining internal junctions which are not paired with an adjacent internal junction are paired with an arbitrary non-internal junction. The previously red external bond on that junction is re-coloured white and the common internal bond is then coloured red. The colouring described meets the requirements of the definition for a pseudo-base colouring.

For a given bond graph  $B$ , there are usually many different possible pseudo-base colourings. We illustrate this with the following example.

*Example 3.* Pseudo-base colourings, of the type constructed in the proof of Theorem VI, are shown in Fig. 3. The colouring of the bond graph of Fig. 3(a) is

as described for a bond graph without internal junctions. Figure 3(b) shows such a pseudo-base colouring for a bond graph with internal junctions. We also show, in Fig. 3(c), an alternative pseudo-base colouring for the first of these bond graphs, which is not as constructed in the proof of the theorem.

*Definition.* Suppose that we have a particular pseudo-base colouring of a bond graph  $B$ . Consider the collection of external bonds, denoted by  $\beta$ , which consists of any red external bond incident on a  $p$ -junction of  $B$  and any white external bond incident on an  $s$ -junction of  $B$ . This set of bonds will be called the *pseudo-base set* corresponding to the pseudo-base colouring given for  $B$ .

*Notation.* We defined above the symbols  $e_s$  and  $e_p$ . We extend this notation by using the subscripts  $r$  and  $w$  to denote red and white bonds respectively. This defines the symbols below :

- $e_{sr}$  = the number of red external bonds on all  $s$ -junctions,
- $e_{pr}$  = the number of red external bonds on all  $p$ -junctions,
- $e_{sw}$  = the number of white external bonds on all  $s$ -junctions,
- $e_{pw}$  = the number of white external bonds on all  $p$ -junctions.

The junction count symbols  $s(B)$  and  $p(B)$  will also be modified by the subscripts  $e$ ,  $i$ ,  $r$  and  $w$ . We define the following symbols :

- $s_{er}$  = the number of  $s$ -junctions with a red external bond,
- $s_{ir}$  = the number of  $s$ -junctions with a red internal bond,
- $p_{er}$  = the number of  $p$ -junctions with a red external bond,
- $p_{ir}$  = the number of  $p$ -junctions with a red internal bond.

It is important to note clearly that  $e$  and  $i$  denote numbers of *bonds* and  $s$  and  $p$  denote numbers of *junctions*, and that each of these can have modifying subscripts. The combinatorial arguments used in the sequel are quite involved and this notation simplifies the work considerably.

*Lemma.* For a proper simple bond graph the following are valid :

- (i)  $s_{er} = e_{sr}$ ,
- (ii)  $p_{er} = e_{pr}$ ,
- (iii)  $p_{ir} = s_{ir}$ ,
- (iv)  $p = p_{er} + p_{ir}$ ,
- (v)  $s = s_{er} + s_{ir}$ .

*Proof:* The first two results follow because a junction can have a maximum of one red external bond. If this is the case the bond contributes to both number counts. A red internal bond contributes to neither of the counts. Such a bond will be a red internal bond on both an  $s$ - and a  $p$ -junction, the (internal) bond joining these two junctions. Thus it contributes to both counts in (iii). Finally, the last two follow since a red bond must be either internal or external.

### Theorem VII

Let  $B$  be a simple proper bond graph. For any pseudo-base colouring of  $B$  the number of bonds in the corresponding pseudo-base set is given by

$$\# \beta = p - s + e_s.$$

*Proof:* From the definition of  $\beta$ , and using the lemma above, we have

$$\begin{aligned} \# \beta &= e_{pr} + e_{sw} \\ &= e_{pr} + (e_s - e_{sr}) \\ &= p_{er} + e_s - s_{er} \\ &= (p - p_{ir}) + e_s - (s - s_{ir}) \\ &= p - s + e_s + (s_{ir} - p_{ir}) \\ &= p - s + e_s. \end{aligned}$$

*Definition.* Let  $\beta$  be a pseudo-base set for a bond graph  $B$ . Then  $\beta$  is a set of external bonds of  $B$ , say  $\beta = \beta_1 \beta_2 \dots \beta_r$ , where  $r = p - s + e_s$  from Theorem VII. Consider the set  $\hat{\beta} = \{\beta_1, \beta_2, \dots, \beta_r\}$ . This is a set of  $r$  vectors in  $W_e(B)$ , the external bond space of  $B$ , each of which consists of one external bond. The subspace spanned by  $\hat{\beta}$  will be denoted by  $W_\beta(B)$  or simply  $W_\beta$ . This space is defined for a bond graph  $B$  with respect to a chosen pseudo-base set for  $B$ .

*Theorem VIII*

Let  $B$  be a proper simple bond graph with pseudo-base set  $\beta = \beta_1 \beta_2 \dots \beta_r$ , where each  $\beta_k$  is a single (external) bond of  $B$ . The space  $W_\beta$  is a subspace of  $W_e$  of dimension  $r = p - s + e_s$ . The set  $\hat{\beta}$  is a basis for  $W_\beta$ .

*Proof:* The results are elementary consequences of the definitions [see the discussion of the bond vector space in (I), in particular Theorem III].

*Definition.* Let  $B$  be a bond graph with a pseudo-base colouring defined. A red/white loop in  $B$  is a circuit of alternately red and white internal bonds of the underlying graph of  $B$ . We say that the pseudo-base colouring contains a red/white loop.

*Theorem IX*

Let  $B$  be a proper simple bond graph. If  $\beta$  is a pseudo-base set for  $B$ , defined by a pseudo-base colouring which contains no red/white loop, then

$$W_s \cap W_\beta = \{\emptyset\}.$$

*Proof:* Suppose that we have a non-empty set  $\alpha \in W_s \cap W_\beta$ . Then  $\alpha$  is a linear combination of vectors in  $\hat{\beta}$  and also a linear combination of the elementary junction vectors which form a basis of  $W_s$ . Thus  $\alpha$  is a set of external bonds, a subset of the pseudo-base set  $\beta$ , which can be written as a sum of elementary junction vectors of  $B$ .

Choose one of the (external) bonds in  $\alpha$ , say  $\beta_k$ . Suppose that this  $\beta_k$  is incident on  $s$ -junction  $s_1$ . (The case of a  $p$ -junction will be considered later.) We construct

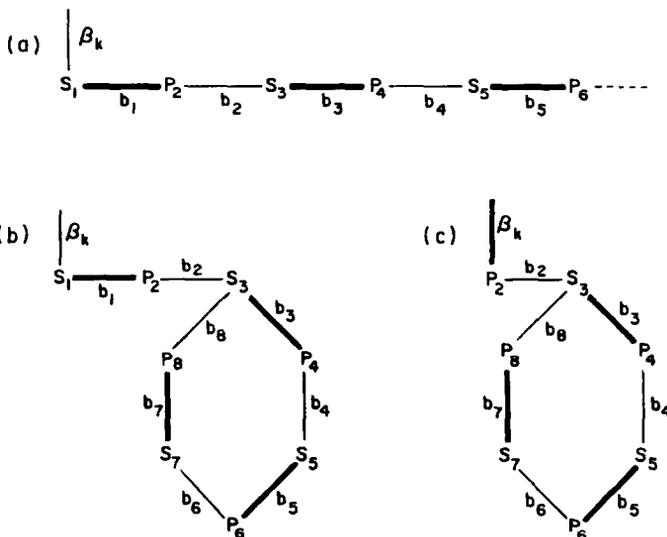


FIG. 4. Illustrations for proof of Theorem IX. (a) Construction of red/white internal path ; (b) example of necessary red/white loop ; (c) the case where  $\beta_k$  is on a  $p$ -junction.

a path of adjacent junctions, beginning with  $s_1$ , which must eventually close onto itself. The argument is illustrated by the diagrams shown in Fig. 4.

Since  $\beta_k \in \alpha$ , the elementary junction vector on junction  $s_1$  must be included in the elementary decomposition of  $\alpha$ . This means that all the other external bonds on  $s_1$  must be included in  $\alpha$ . But  $\alpha$  is a subset of the pseudo-base set and so all of the external bonds on  $s_1$  must be coloured white. In particular, there must be a red internal bond, say  $b_1$ , incident on  $s_1$ .

Let  $p_2$  be the  $p$ -junction, adjacent to  $s_1$ , on which  $b_1$  is incident. Now  $b_1 \notin \alpha$ , since  $\alpha$  contains no internal bond. Thus the elementary decomposition of  $\alpha$  must include a  $p$ -type vector on junction  $p_2$ , of form  $b_1 b_2$ , where  $b_2 \neq b_1$  is incident on  $p_2$ . Since  $b_1$  is a red bond, there can be no red external bond on  $p_2$  and so no external bond on  $p_2$  is included in  $\alpha$ . Now we may assume that  $b_2$  is an internal bond. Otherwise  $b_2$  would appear in a second  $p$ -type vector on  $p_2$ , so that  $b_2$  is not included in  $\alpha$ . We have shown that  $b_2$  is an internal white bond incident on  $p_2$ .

Let  $s_3$  be the  $s$ -junction, adjacent to  $p_2$ , on which  $b_2$  is incident. Since  $b_2$  is internal it is not included in  $\alpha$ , and so the  $s$ -type elementary junction vector on  $s_3$  must appear in the elementary decomposition of  $\alpha$ . Then all the external bonds on  $s_3$  must be in  $\alpha$  and so none of these may be a red bond. However  $b_2$  is a white bond. Therefore there must be a red internal bond,  $b_3 \neq b_2$ , incident on  $s_3$ .

We can repeat the argument above, replacing  $b_1$  with  $b_3$ . This reasoning can be repeated continuously, thus constructing a path of alternately red and white internal bonds, as shown in the example of Fig. 4(a). Since there are a finite number of junctions in  $B$ , the path must be finite. The only way in which this can occur is if one of the even numbered (white) bonds is incident on an odd-numbered  $s$ -junction which appears earlier in the path. An example of this situation is shown in Fig.

4(b). In particular, we must necessarily have a circuit of alternately red and white internal bonds.

If the original external bond  $\beta_k$  is incident on a  $p$ -junction, the argument above applies with a minor change, as illustrated in Fig. 4(c).

We have shown that the pseudo-base colouring which defines  $\beta$  must contain a red/white loop when  $W_s \cap W_\beta$  contains a non-empty set, which proves the theorem.

*Corollary.* Let  $B$  be a proper simple bond graph. If  $\beta$  is a pseudo-base set for  $B$ , defined by a pseudo-base colouring which contains no red/white loop, then

$$W_{cy} \cap W_\beta = \{\emptyset\},$$

where  $W_{cy}$  is the cycle space of  $B$ .

The converse of Theorem IX is false, as shown by the following example.

*Example 4.* Consider the bond graph shown in Fig. 5. A pseudo-base colouring with a red/white loop is shown, with red bonds being indicated by heavy lines. The pseudo-base set  $\beta = 135$ . This gives a basis  $\hat{\beta} = \{1, 3, 5\}$  for  $W_\beta$ , and so we have

$$W_\beta = \{\emptyset, 1, 3, 5, 13, 15, 35, 135\}.$$

It can easily be checked that none of these vectors is  $s$ -equivalent to  $\emptyset$  and so, by Theorem III of (2),  $W_\beta \cap W_s = \{\emptyset\}$ .

*Theorem X*

Let  $B$  be a proper simple bond graph and let  $\beta$  be a pseudo-base set defined by a pseudo-base colouring of  $B$  which contains no red/white loop. Then we have the following direct sum decomposition of the bond space of  $B$ :

$$W = W_s \oplus W_\beta.$$

*Proof:*  $W_s$  and  $W_\beta$  are subspaces of the bond space  $W$ . The proof of Theorem IV shows that

$$\dim W_s = s - p - e_s + \dim W$$

and Theorem VIII shows that

$$\dim W_\beta = p - s + e_s.$$

Thus  $W_s$  and  $W_\beta$  are subspaces of  $W$  whose dimensions add to  $\dim W$  and, by

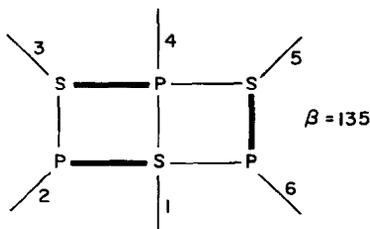


FIG. 5. Counter-example to converse of Theorem IX.

Theorem IX these subspaces have null intersection. The result of the theorem follows.

*Corollary.* Let  $B$  be a proper simple bond graph. If  $\beta$  is a pseudo-base set, defined by a pseudo-base colouring of  $B$  which contains no red/white loop, then

$$W_\beta \approx W/W_s.$$

*Proof:* The result follows from Theorem V;  $W_\beta$  is the subspace  $Y$  of that theorem.

*Theorem XI*

Let  $B$  be a proper simple bond graph and let  $\beta$  be a pseudo-base set defined by a pseudo-base colouring of  $B$  which contains no red/white loop. Then  $\beta$  is a base of  $M(B)$ , the cycle matroid of  $B$ .

*Proof:* First, noting that  $W_\beta$  is a subspace of  $W_e$ , we conclude, applying the result of Theorem IX that  $W = W_\beta + W_s$  is a subspace of  $W_e + W_s$ . Hence we have  $W = W_e + W_s$ . Now the isomorphism

$$(W_e + W_s)/W_s \approx W_e/(W_s \cap W_e)$$

is a standard result concerning quotient spaces (10). The left side is  $W/W_s$  and the right side is  $W_e/W_{cy}$ . By applying the corollary to Theorem X, we conclude that

$$W_\beta \approx W_e/W_{cy}.$$

We now establish the connection with the cycle matroid of  $B$ . We use the notation of (3), which also describes the construction of the cycle matroid of a bond graph. Recall that  $E_e$  denotes the external bond set of  $B$ . Now  $L(E_e)/N_{cy}$ , when regarded as a vector space, is isomorphic to  $W_e/W_{cy}$ . According to Theorem IV of (3), the matroid induced on  $L(E_e)/N_{cy}$  by linear independence is isomorphic to the matroid  $M(N_{cy})$ , which, by definition, is the cycle matroid of  $B$ ,  $M(B)$ . Combining these results, we conclude that the matroid induced on  $W_\beta$  by linear independence is isomorphic to the cycle matroid of  $B$ . In particular, this implies that the basis  $\hat{\beta}$  of  $W_\beta$  corresponds to a base  $\beta$  of  $M(B)$ .

*Corollary.* Let  $B$  be a proper simple bond graph and let  $\beta$  be a pseudo-base set defined by a pseudo-base colouring of  $B$  which contains no red/white loop. Then the external bond space has the following direct sum decomposition:

$$W_e = W_{cy} \oplus W_\beta.$$

*Proof:* We use the result  $W_\beta \approx W_e/W_{cy}$ , from the proof of the theorem, to conclude that

$$\dim W_{cy} + \dim W_\beta = \dim W_e.$$

The corollary follows, by combining this formula with the corollary to Theorem IX.

*Definition.* If a pseudo-base set,  $\beta$ , of a bond graph  $B$ , corresponds to a base of the cycle matroid of  $B$ , we say the pseudo-base colouring is a *base-colouring* of  $B$  and  $\beta$  is a *base set* for  $B$ .

The result of Theorem XI says that a pseudo-base colouring defines a base-set,  $\beta$ , when it contains no red/white loop, but Example 4 will provide a counter-example to show that the converse is false. We discuss this example below (Example 7) after we establish the connection between base-colourings and causality.

It should be apparent that base colourings and base sets for a bond graph provide a convenient means of determining bases for the cycle matroid and its dual, the co-cycle matroid, of the bond graph. Nevertheless, regardless of any consideration of base colourings, the combinatorial results of this section provide formulae for the ranks of these matroids. In particular, the formulae are valid whether or not a base colouring exists. A vector space version of the result provides formulae for the dimensions of the cycle and co-cycle spaces of a bond graph.

*Definition.* The rank of a bond graph  $B$ , denoted by  $r(B)$ , is the rank of the cycle matroid of  $B$ ,  $M(B)$ . Dually the co-rank of a bond graph  $B$ , denoted by  $q(B)$ , is the co-rank of  $M(B)$  or, equivalently, the rank of the co-cycle matroid of  $B$ ,  $M^*(B)$ .

*Theorem XII*

Let  $B$  be a proper simple bond graph. We have the dual formulae

$$r(B) = p - s + e_s$$

and

$$q(B) = s - p + e_p.$$

*Proof:* From the proof of Theorem XI, we see that the rank of  $M(B)$  equals the dimension of  $W/W_s$ , regardless of any properties of  $W_\beta$ . From Theorem IV this is  $p - s + e_s$ . Duality follows from Theorem IV\*.

*Theorem XIII*

Let  $B$  be a proper simple bond graph. For the dimensions of the cycle and co-cycle spaces of  $B$  we have the dual formulae

$$\dim W_{cy} = s - p + e_p = q(B)$$

and

$$\dim W_{co} = p - s + e_s = r(B).$$

*Proof:* From the proof of Theorem XI we have  $W/W_s \approx W_e/W_{cy}$  and so these must have the same dimension. By Theorem IV we have

$$\dim W/W_s = p - s + e_s$$

and also we know that

$$\begin{aligned} \dim W_e/W_{cy} &= \dim W_e - \dim W_{cy} \\ &= e - \dim W_{cy}. \end{aligned}$$

Combining these gives

$$\begin{aligned} \dim W_{cy} &= s - p + e - e_s \\ &= s - p + e_p. \end{aligned}$$

The second formula follows by duality.

*Example 5.* Consider the bond graph and base-colouring shown in Fig 6(a). The corresponding base set is  $\beta = 13$ .  $W_\beta$  is spanned by the basis  $\hat{\beta} = \{1, 3\}$  and so

$$W_\beta = \{\emptyset, 1, 3, 13\}.$$

The  $s$ -space  $W_s$  is spanned by the basis of elementary junction vectors  $\{12o_1, o_13, 34\}$  and this gives

$$W_s = \{\emptyset, 123, 124, 34, 12o_1, o_13, o_14, 12o_134\}$$

with cycle space

$$W_{cy} = \{\emptyset, 123, 124, 34\}$$

spanned by the basis  $\{124, 34\}$ . It is clear that the results of the above theorems are satisfied by this bond graph and base set.

In Fig. 6(b) we show an associated graph for this bond graph. The choice of base  $\beta = 13$  for  $M(B)$  corresponds to choosing the spanning tree  $T(1, 3)$  in the associated graph.

### V. The Connection with Causality

Choosing a base colouring for a bond graph  $B$  is, in fact, equivalent to the standard technique called *selection of causality* in physical system modelling by bond graphs. Instead of colouring the distinguished bond red, a *causal stroke* is placed on one end of each bond in  $B$ . We allow precisely one causal stroke to be on a  $p$ -junction and precisely one bond end without a causal stroke to be on an  $s$ -junction. This clearly identifies exactly one distinguished bond on each junction of  $B$ , which is the requirement for a pseudo-base colouring. A base-colouring corresponds to a valid assignment of causality. If the choice of causal strokes leads to an invalid assignment of causality the colouring is a pseudo-base colouring and not a base-colouring.

The external bonds which have the causal stroke on the junction end are called *junction causal*. The pseudo-base set  $\beta$  consists of all the junction causal bonds. A red/white loop is called a *causal loop*.

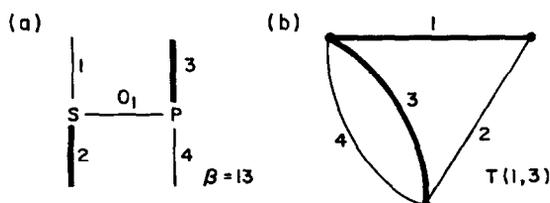


FIG. 6. Illustrations for Example 5. (a) Bond graph with base colouring ; (b) associated linear graph with corresponding spanning tree.

We illustrate the definitions above, in Fig. 7, by showing the causal stroke notation for the choice of base  $\beta = 13$  given by the base-colouring of the bond graph in Fig. 6(a).

*Remark 4.* The result of Theorem XI is the theoretical justification for the procedure of causality selection in the bond graph literature. It is clear that either selection of causality on a bond graph  $B$  or choosing a spanning tree on an associated graph  $G(B)$  is equivalent to choosing a base for a matroid isomorphic to the cycle matroid of the bond graph,  $M(B)$ , or the cycle matroid of the graph,  $M(G)$ .

*Remark 5.* For computer applications it is possible that the base-colouring method for indicating causality may be superior to the causal stroke method. The storage requirements for recording one distinguished bond on each junction of a bond graph are less than the corresponding storage for a causal stroke on every bond. Also, manipulation of causality is simpler in the base colouring notation.

*Remark 6.* It is often stated in the bond graph literature that the addition of causal strokes to a bond graph is an orientation of the bonds. For instance, Perelson and Oster (11) state that ‘‘causally directing a bond graph provides an orientation for each edge’’. This is a rather superficial interpretation of the notation, since it is merely the underlying graph of the bond graph which is being oriented. The causal strokes do not provide any orientation to the combinatorial structure which models the structure of the system. An *orientation* of a matroid has a precise combinatorial meaning which we will discuss in (4). An oriented matroid can be obtained from either a linear graph or bond graph diagram of the matroid by a simple procedure. In the case of a linear graph, we always obtain an orientation by directing the edges of the graph. The addition of half-arrows to the bonds of a bond graph *may* define an orientation of the matroid, provided certain conditions are satisfied. When illustrated by colouring the bonds as a base-colouring, the selection of causality loses its superficial appearance of an orientation, and appears to be related to the method of indicating a spanning tree on a linear graph, to which we have seen that it is equivalent.

*Example 6.* Consider the bond graph,  $B$ , shown in Fig. 8(a) with a choice of base colouring corresponding to  $\beta = 1\ 2\ 6\ 8\ 10\ 11$ . We give the causal stroke notation for this choice of  $\beta$  in Fig. 8(b). An associated graph  $G(B)$  is given in Fig. 8(c) and we can see that  $\beta$  is the corresponding spanning tree for  $G(B)$ .

*Example 7.* Consider the bond graph,  $B$ , of Fig. 5 which was discussed in Example 4. We show an associated graph for  $B$  in Fig. 9(a), with the spanning tree corresponding to the choice of  $\beta = 135$ . The pseudo-base set is a base set, in spite of the fact that  $\beta$  contains two red/white loops. The causal stroke notation for this

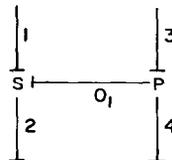


FIG. 7. Causal stroke notation for base-coloured bond graph of Fig. 6(a).

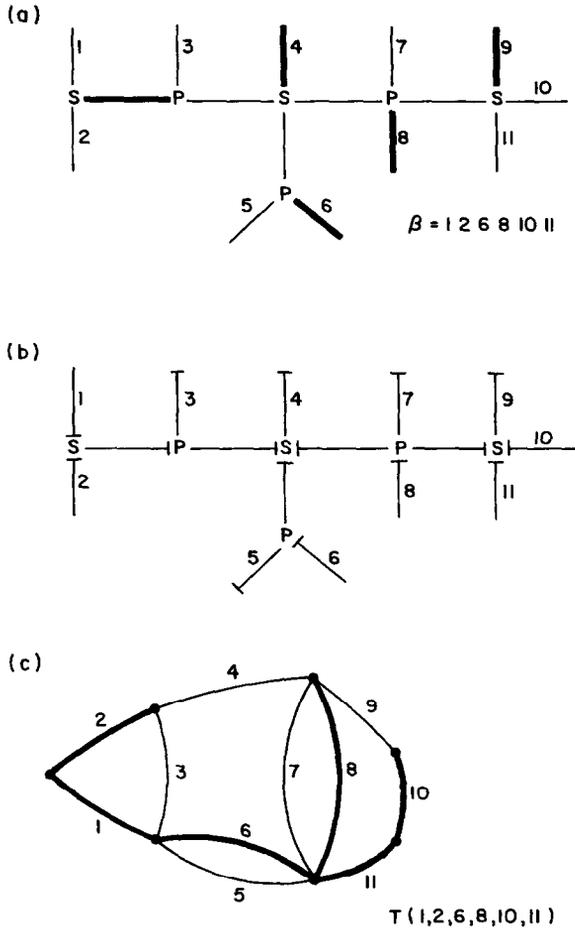


FIG. 8. Illustrations for Example 6. (a) Bond graph with base colouring; (b) base colouring indicated with causal stroke notation; (c) equivalent linear graph with corresponding spanning tree.

choice of base is given in Fig. 9(b). This shows a bond graph with two causal loops, corresponding to a valid selection of causality.

*Remark 7.* Example 7 shows that a causal loop does not always lead to an invalid assignment of causality, contrary to a customarily accepted fact expressed, for instance, in (6, 7, 11). In (7), the authors state that “a causal assignment containing causal loops is not an allowable causal assignment”. In the comments to Theorem 2 of (6) it is stated that “[for a causal loop] the branches which correspond to the junction causal bonds no longer form a tree”. The example above shows that this is not always the case.

*Remark 8.* We shall discuss the theorems of (6) further, in Part V (4), when we consider directed bond graphs and orientation, in particular the case of an odd

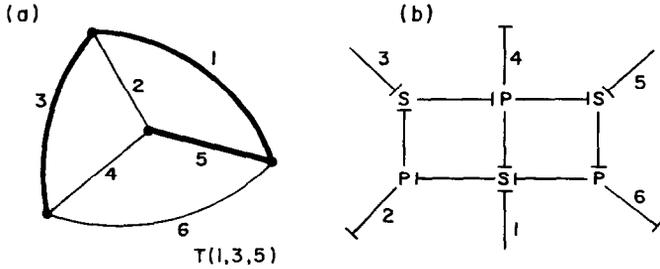


FIG. 9. Counter-example to converse of Theorem XI. (a) Associated linear graph and corresponding spanning tree; (b) bond graph with causal loop and valid causality assignment.

loop. An *odd loop* is a loop with an odd number of internal bonds directed (by half-arrows) in each direction around the loop. It is interesting to consider the ‘pathological example’ provided there [Fig. 4 of (6)] of a bond graph for which every assignment of causality leads to a causal loop. In fact, for this bond graph, any choice of causal strokes corresponds to a valid assignment of causality! An example is shown in Fig. 10.

*Remark 9.* The properties of junction structure matrices, such as the ranks of the effort and flow matrices, are consequences of our theorems concerning bond graph matroids. For instance, (non-oriented versions of) the first six theorems of

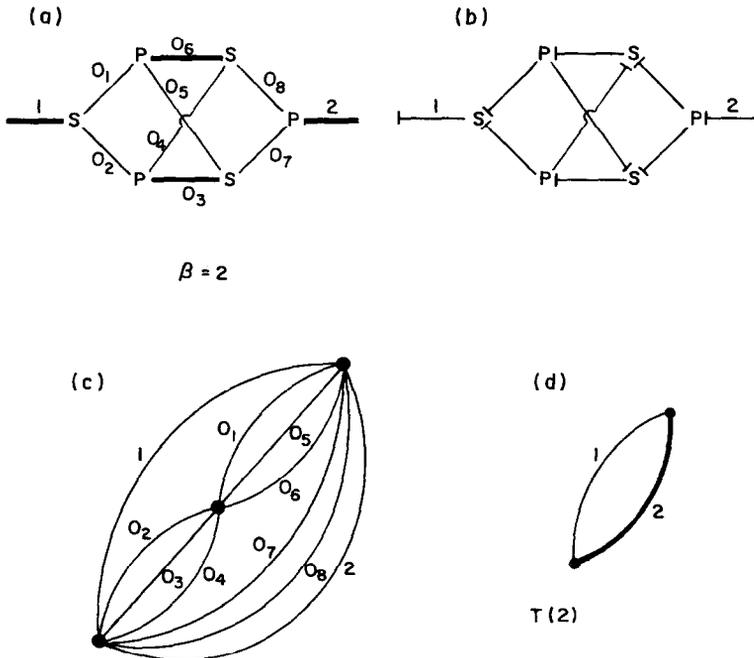


FIG. 10. The ‘‘pathological bond graph’’ of Perelson. (a) Base-coloured bond graph includes a red/white loop; (b) causal stroke notation for (a) includes a causal loop; (c) augmented graph; (d) associated graph with correct corresponding spanning tree.

(5) follow from our Theorem XII or XIII. Combining our theorems with the orthogonality theorem of Part II (2), we obtain Theorem 7 of (5). [We should note that Theorems 3 and 6 of (5) are incorrect as stated in that paper. The correct theorems were used in a subsequent paper by these authors (7), and there was also a discussion of the error in (12).] We shall discuss oriented versions of these theorems in Part V (4).

## **VI. Conclusions**

Matrix representations of bond graph matroids have been examined and using this concept we have provided a rigorous proof of the mathematical equivalence of the linear graph modelling methods. All equivalent bond graphs and associated linear graphs have the same matroid structures and, since it is a matrix representation of these which is used to write system structural equations (generalized Kirchhoff equations), we see that the methods must be identical.

The procedure of causality selection has been analysed. For convenience, we showed that this is equivalent to choosing a distinguished bond on each junction of the bond graph and colouring it red, say. This determines a pseudo-base colouring, which is used to define a corresponding pseudo-base set (the usual junction causal external bonds).

If the colouring contains no red/white loop of internal bonds (causal loop) the pseudo-base set is a base for the cycle matroid of  $B$ . However the converse is false, since we give an example of a bond graph with a causal loop for which causality is valid. Choosing a base of the cycle matroid corresponds to a valid selection of causality.

Various combinatorial formulae are proved concerning ranks and dimensions. Formulae for the ranks of effort and flow matrices and the number of independent equations obtained from a bond graph are consequences of our results.

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