

The Mathematical Foundations of Bond Graphs—III. Matroid Theory

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ABSTRACT: *The cycle and co-cycle matroids of a bond graph are defined using chain group matroids derived from the cycle and co-cycle vector spaces of a bond graph. The relationship between these structures is investigated and various results are proved. A precise equivalence is defined for bond graphs. Duality theory is seen to be very clear in the context of bond graph matroids.*

I. Introduction

Many techniques are available for the mathematical modelling of a physical system. Some of the methods, such as Lagrangian mechanics, may be called *energy methods*, since they use conservation of energy explicitly and a minimum of system structure, to derive system equations. For discrete physical systems and discretized continuous systems it is also possible to apply methods which use system structure explicitly. In these methods, which may be called *combinatorial methods*, conservation of energy is implied, rather than being used explicitly.

There are currently two popular combinatorial methods: *linear graph* and *bond graph* methods. The engineering community seems to be divided on the question of which technique to employ. Unfortunately, there is little interaction between the two groups of researchers. This division is curious, since the two methods are intimately connected. In the area of common ground, in particular derivation of state equations, the methods can be put into an exact parallel. This fact has been clear since the invention of bonds graphs; a theoretical explanation was given in (1, 2). A linear graph and a bond graph are simply different pictorial representations of the same combinatorial information, the interconnection topology of the components, which we call the *system structure*.

In the electrical domain, graph theory has been used in the analysis of networks since Kirchoff's work *ca* 1850. The generality of the graph theory approach in all physical domains was first recognized by Trent (3). A well-established technique now exists and is described in many textbooks, for instance Koenig *et al.* (4). There exists a substantial combinatorial theory on which the method is based: the theory of graphs. A linear graph becomes a *system graph* when it is used in a system model.

The second combinatorial method was devised by Paynter (5), who first introduced the concept of bond graph. Information concerning the topology of the

system is coded into the symbols which appear in the bond graph. Several texts have appeared describing the fundamentals of bond graph modelling, for instance, Rosenberg and Karnopp (6).

For a bond graph model, there is no combinatorial foundation analogous to the graph theoretical basis for system graph modelling. Bond graphs have always been used in an *ad hoc* manner. This flows naturally from applied engineering procedures (even textbooks), where basic knowledge of physics is applied, *ad hoc*, to a machine, electrical component and/or circuit. Bond graphs for such systems are constructed from engineering drawings, schematics or circuits with pictorial symbols.

This paper is a continuation of (1, 2), in which an algebraic combinatorial theory of bond graphs is introduced. These two papers are Parts I and II on the mathematical foundations of bond graphs. In this series, we consider a bond graph as a combinatorial structure, which we call a *combinatorial bond graph*, and develop a mathematical theory describing its properties. The connection with bond graph modelling is established when we consider a combinatorial bond graph as a *junction structure* (see 7) in the formation of a model of a physical system. We call the resulting model a *system bond graph*. This process exactly parallels the formation of a system graph, which models the physical system, from a linear graph, which contains the combinatorial structure of the system.

In the first two papers we use elementary mathematical concepts from linear algebra, to define vector spaces associated with a bond graph. The combinatorial information which the bond graph diagram is to encode, is reflected in the algebraic properties of these vector spaces. We shall use the terminology and notation of the two earlier papers freely in this paper. For convenience, we provide a short summary below, however (1, 2) may be referred to for the detailed definitions and results.

In Part I we defined precisely a *non-directed combinatorial bond graph*. We can illustrate this definition by considering a diagram of a bond graph junction structure which contains only junctions (we use *s*- and *p*-junctions for reasons explained in Part I), no power half-arrows, no causal strokes and no physical components, the bond graph we define carries no structure *a priori*. The customary meaning of the bond graph notation is provided by the combinatorial structure which is given to the bond graph via the definition of its vector spaces.

The principal algebraic device which we use is the *symmetric difference* of two sets *X* and *Y*. This is defined to be the set which consists of all the elements in *X* or *Y* which are not in both *X* and *Y*. The symmetric difference of *X* and *Y* is denoted by $X + Y$. For example,

$$\{1, 2, 4, 5\} + \{1, 3, 4\} = \{2, 3, 5\}.$$

Consider all possible sets of bonds of a bond graph *B*. This collection of sets, together with the operation of combining them by symmetric differences, is a vector space over $GF(2)$, the field of 01 arithmetic. This is called the *bond space* of *B*.

The notation for collections of sets is rather inconvenient. Therefore we adopt a simplified notation for a set in which we write the elements side by side, without commas or braces. For instance, we denote the set $\{1, 2, 4, 5, 7\}$ by 12457 and the set of sets $\{\{123\}, \{124\}, \{34\}\}$ is denoted simply by $\{123, 124, 34\}$. The empty set

$\{ \}$ is denoted by \emptyset and the number of elements in a set A is denoted by $\#A$. If A and B are sets then $A - B$ means the set of elements in A but not in B .

We can construct two subspaces of the bond space of a bond graph B . First we define fundamental vectors which come directly from the bond graph diagram. An *elementary junction vector* is any set of bonds which consists of either :

- (i) all the bonds incident on some s -junction (s -type), or
- (ii) any two bonds incident on the same p -junction (p -type).

The space spanned by the elementary junction vectors of B is a subspace of the bond space, called the s -space of B .

The s -space is too large to provide the required algebraic structure for a bond graph. The *cycle space* of B is the subspace of the s -space which consists of vectors containing no internal bond. The algebraic structure provided for B by its cycle space contains the combinatorial information which is used implicitly in every system bond graph model. The physical components are associated with the external bonds of a bond graph and the cycle space models the interconnections of the system components.

The *dual bond graph* of a bond graph B , denoted by B^* , is the bond graph obtained by dualizing the combinatorial structure given by B . All of the algebraic structure defined above can be applied to the bond graph B^* . With respect to the original bond graph B we call these structures *dual structures* and denote them by the same names with a 'co-' prefix. For instance, the co-cycles of B are the cycles of B^* . A complete analysis of duality and the relation between structures and dual structures is contained in (2).

The algebraic results of Parts I and II can be made most transparent if we use matroids for the combinatorial model. In this paper, and (8, 9), we explore the combinatorial structure of bond graphs in terms of matroid theory. By this further abstraction, we obtain a theory which is, in fact, simpler and, in the view of the authors, more elegant. We define matroids associated with a bond graph, which we call the cycle and co-cycle matroids of the bond graph.

Matroids have appeared in the engineering literature. For instance, Swamy and Thulasiraman (10) develop matroid theory and apply it in applications concerning efficiency of certain algorithms used in network analysis. Most of the engineering applications of matroids seen to be in such an optimization context, where the "greedy algorithm," described in (11), is applied. The use of matroids to model structure seems to be a new application to engineering.

A bond graph is inextricably linked to its matroids. Some connection between bond graphs and matroids has been suspected, as remarked by Perelson and Oster (12). We show here that a bond graph is a concise planar diagram of a specific type of matroid, known as a binary matroid. All such matroids have a bond graph representation, including the so-called non-graphic binary matroids which are represented by non-graphic bond graphs. (A non-graphic bond graph is defined in (1) and we give an equivalent definition in this paper in terms of matroids.) Thus a bond graph is a new pictorial diagram for any binary matroid, a result which may be interesting to combinatorial mathematicians.

The extremely close connection between matroids and bond graphs is exemplified by considering duality. The elegant and symmetric duality theory for matroids, is reflected in similar properties of duality theory for bond graphs, as discussed in (2). We elaborate on this point in this paper. The authors recognize that many readers will be unfamiliar with matroid theory. Therefore in Section II we summarize the basic definitions and theory. Our approach to defining bond graph matroids is via the algebraic theory of the first two papers. In particular, we use the vector spaces associated with a bond graph to define the matroids.

The most convenient way to make these definitions is to use chain group matroids developed by Tutte (13). A bond graph can be constructed directly from a chain group. Also, we will have available many useful results connecting vector spaces, chain groups and their associated matroids, precisely what we shall need in our proofs. Therefore, we have included an introduction to the theory of chain groups in Section II. This should be sufficient to summarize the definitions and theory and to state the results we use.

After introducing matroid theory, we begin the discussion of bond graph matroids in Section III. A precise definition of the meaning of equivalence of bond graphs is given in Section IV. Several examples are given to illustrate this concept. We discuss the implications for bond graph modelling: distinct system bond graph models are equivalent when their combinatorial bond graphs are equivalent.

The results of Section V provide a complete and precise mathematical statement describing the connections between linear graphs and combinatorial bond graphs, as pictorial representations of combinatorial structure. We provide examples in Section VI, a discussion of the significance of some of the results to physical system modelling in Section VII and conclude with a summary in Section VIII.

We provided a summary of the theory of graphs in (1), and a discussion of duality for graphs in (2). The reader may consult these for definitions of terminology and as a general reference.

II. Matroids and Chain Groups

In this section, we introduce matroid theory and the theory of chain groups. We give definitions and state results without proof here. For proofs of the results stated, and a complete treatment of these subjects refer to (11). For a different and more intuitive approach to this subject, an excellent reference is (14), where a theory of "graphoids" is developed. These are generalized graphs and are defined algebraically as a structure consisting of a matroid and its dual matroid. This approach to the theory is parallel to what we require, but the results are stated in a way which is unnatural to apply in the bond graph context. Nevertheless the reader may prefer to consult (14) before reading our more abstract approach to the theory. It is interesting to note that a bond graph is really a diagram of a graphoid.

2.1. Introduction to Matroid Theory

A *matroid* is a set, together with a distinguished collection of its subsets, called the *independent sets* of the matroid. The independent sets satisfy certain axioms,

which were originally motivated by the properties of linear independence in vector spaces. The origin of the subject of matroid theory is the classic paper of Whitney (15). The author introduces the subject as the “abstract theory of linear dependence”, which illustrates the motivation.

There are several equivalent sets of axioms used to define a matroid, each set being natural in certain applications. It is often remarked that a matroid has an unusually large number of equivalent axiom systems. We state the *independence axioms*, which define a matroid by specifying its independent sets.

Definition. A *matroid* is a finite set S , called the *underlying set*, and a class of subsets of S , called the *independent sets* of the matroid, such that the following axioms are satisfied:

- (1) The empty set \emptyset , is independent.
- (2) Any subset of an independent set is also independent.
- (3) If X and Y are independent sets and $\#X = \#Y + 1$ then there exists an $x \in X - Y$ so that $Y \cup \{x\}$ is an independent set.

A subset of S which is not independent is called a *dependent set*.

Definition. A *base* of a matroid M is a maximal independent set. A minimal dependent set is called a *circuit* of M .

It can be shown that two bases of a matroid have the same number of elements. There are also equivalent axiom systems which define a matroid in terms of its circuits or its bases. Thus a matroid can be defined by specifying its collection of circuits or bases.

Consider a vector space V and let S be any finite subset of vectors. We say that a subset of S is independent if its vectors are linearly independent vectors in V . These independent sets are the independent sets of a matroid on S . Axioms (1) and (2) are trivial facts of linear algebra and axiom (3) is known as the Steinitz exchange property. We say that the matroid structure is *induced* on S by *linear independence*. Any matroid isomorphic to a matroid of this type is called a *vectorial matroid*.

Definition. Two matroids M_1 and M_2 are *isomorphic* if there is a bijection between their underlying sets which preserves independence. Equivalently we could say that the bijection must preserve the bases or the circuits of the matroids. We write $M_1 \approx M_2$ if M_1 and M_2 are isomorphic matroids.

We give next another simple example which comes from linear algebra. Let A be an $n \times m$ matrix with entries in a field F . Let S be the index set $\{1, 2, \dots, m\}$. The columns of A are vectors in the vector space F^n , the vector space of n -tuples with entries in F . We can define a matroid on the set S by calling a subset of S independent if the corresponding columns of A are linearly independent in F^n . A *matrix matroid* is a matroid which is isomorphic to the matroid induced on the columns of a matrix by linear independence.

Definition. If M is a matrix matroid with respect to the matrix A with entries in the field F , then we say M is *representable over F* and A is a *matrix representation* of M .

Remark 1. We discuss matrix representations in detail in (8). The concept of matrix representation is central to the application of bond graphs to physical

system modelling. It is a matrix representation which is used when a cutset or circuit matrix is produced from a system graph or a junction structure matrix is formed from a system bond graph model. Even if these matrices are not explicitly used such representations are implicit whenever generalized Kirchhoff equations are written for a linear graph model or junction equations are written for a bond graph model.

Definition. Let T be some subset of the underlying set, S , of a matroid M . The *rank* of T is the number of elements in a maximal independent set which is a subset of T . The *rank of the matroid* M , denoted by $r(M)$, is the rank of the set S , or equivalently the number of elements in any base of M . T is a *spanning set* for M if it contains a base of M .

Definition. Let M be a matroid. We define the *dual matroid*, M^* , of M , by specifying its bases: if β is a base of M then $S - \beta$, the complement of β , is a base of M^* . The fact that M^* is a matroid is easily proved (see **11**). A *co-base* of M is a base of M^* . A *co-circuit* of M is a circuit of M^* . The *co-rank* of M , denoted by $q(M)$, is the rank of M^* .

For matroids, unlike linear graphs, duality is complete and symmetrical. Every matroid has a unique dual matroid, whereas not every graph has a dual graph. Even for a graph G which has a dual graph, there can exist other non-isomorphic graphs which are also dual graphs of G . An example of this is given in (**11**).

Consider a graph G . The collection of cycles and edge-disjoint unions of cycles of G is the class of circuits of a matroid, $M(G)$, on the edge set, E , of G . $M(G)$ is called the *cycle matroid* of the graph G . Dually, the set of co-cycles of a graph G , is the class of circuits of a matroid, $M^*(G)$, on E , called the *co-cycle matroid* of G . The fact that these are matroids follows from the fundamental properties of graphs. The cycle and co-cycle matroids of a graph are dual matroids of each other. A base for the cycle matroid of G is a spanning forest of G , or, if G is connected, a spanning tree of G . The independent sets of $M(G)$ are the forests or trees of G .

Theorem 1

The cycle and co-cycle matroids of a graph G are dual matroids. That is, we have

- (i) $M^*(G) = (M(G))^*$ and
- (ii) $M(G) = (M^*(G))^*$.

Definition. A matroid, M , is *graphic* if it is isomorphic to the cycle matroid of some graph G . If M is isomorphic to the co-cycle matroid of a graph G we say that M is *co-graphic*.

Let G be a graph which has a dual graph. Then $M^*(G)$, the dual matroid of $M(G)$, is isomorphic to $M(G^*)$, where G^* is any abstract dual of G . In particular, if G is planar this implies that all the geometric duals of G have isomorphic cycle matroids, even if the graphs themselves are not isomorphic. This point also illustrates the fact that non-isomorphic graphs can have isomorphic cycle matroids.

2.2. Chain Group Matroids

Definition. Let S be a finite set and F a field. A *chain on S over F* is a map f from S to F . $L(S, F)$, or $L(S)$ if F is clear from the context, will be used to denote

the set of all chains on S over F . The *support* of the chain f is

$$\text{supp}(f) = \{x \in S \mid f(x) \neq 0\}.$$

The *zero chain*, denoted by 0 , maps every element of S to 0 . We define the *sum*, $f+g$, of two chains f and g and the *scalar product*, $c \cdot f$, of $c \in F$ with $f \in L(S)$ by

$$(f+g)(x) = f(x) + g(x)$$

$$(c \cdot f)(x) = c(f(x)).$$

Definition. A set, N , of chains in $L(S)$, is called a *chain group* if N is closed under sums and scalar multiplication.

Definition. A non-zero chain, f , in a chain group N , is an *elementary chain* if there is no chain $g \in N$ so that $\text{supp}(g)$ is properly contained in $\text{supp}(f)$.

It can be shown that the supports of chains in a chain group, N , are the dependent sets of a matroid, $M(N)$, which is called the *matroid associated with* the chain group. In particular, the circuits of $M(N)$ are the supports of the elementary chains of N .

Theorem II

A matroid M on S is isomorphic to the matroid $M(N)$ of the chain group N over F if and only if M is representable over F .

Definition. Let f and g be chains in $L(S)$. We define the *inner product* of f with g by

$$\langle f \mid g \rangle = \sum_{x \in S} f(x) g(x).$$

Definition. If $\langle f \mid g \rangle = 0$ we say the chains f and g are *orthogonal*. Let N be a chain group on S . The *orthogonal complement* of N , N^\perp , is the set of all chains $g \in L(S)$ such that g is orthogonal to every chain in N .

It can be shown that N^\perp is a chain group on S .

Theorem III

Let N be a chain group on S . The dual matroid of $M(N)$ is $M(N^\perp)$.

Finally we note that $L(S)$ can also be regarded as a vector space, a function space of functions from S to F . It is obvious that N is a chain group on S precisely when N is a subspace of $L(S)$.

Theorem IV

If a matroid M is isomorphic to $M(N)$ then M is isomorphic to the matroid induced on the quotient space $L(S)/N$ by linear independence.

Remark 2. In (2), we provided a detailed discussion of the construction of the quotient space of a vector space with a subspace. The result of Theorem IV is crucial in establishing a method of selecting bases for the bond graph matroids we define in the next section. This procedure is central to all the techniques which are used in physical system modelling using bond graphs. The specific technique we refer to is known in the literature as *selection of causality*. In (8) we provide a complete theoretical justification for the procedure.

III. Bond Graph Matroids

Let B be a bond graph. Suppose that $\{v_1, v_2, \dots, v_q\}$ is a basis for $W_{cy}(B)$, the cycle space of B . For each v_k we define a map $f_k: E_c \rightarrow GF(2)$, by

$$\text{supp}(f_k) = v_k,$$

where E_c is the set of external bonds of B and $GF(2)$ is the field with elements $\{0, 1\}$. Thus $f_k(b) = 1$ if $b \in v_k$ and is zero otherwise. Each f_k is a chain on E_c over $GF(2)$. Let N_{cy} denote the chain group generated by the chains defined above. That is, N_{cy} is the set of all chains which are sums of chains of type f_k .

Theorem V

The chain group N_{cy} , when regarded as a vector subspace of $L(E_c)$, is isomorphic to W_{cy} .

Proof: The isomorphism arises from the definition. A basis for N_{cy} is $\{f_1, f_2, \dots, f_q\}$ and the required isomorphism takes f_k to v_k .

Consider $W(B)$, the bond space of B . We shall denote by $W_e(B)$ or W_e the subspace of $W(B)$ which consists of all vectors which contain no internal bond. W_e will be called the *external bond space* of B . Thus W_e is just the vector space defined on the set of all subsets of E_c by symmetric differences. The cycle and co-cycle spaces of B are subspaces of W_e .

Following the same procedure as above, we can define a chain group N_e on E_c over $GF(2)$, by defining a chain f_v for each vector $v \in W_e$. We use the definition

$$\text{supp}(f_v) = v.$$

Thus $f_v(b) = 1$ if $b \in v$ and is zero otherwise.

Theorem VI

N_e is isomorphic to W_e and moreover, we have

$$\langle f_v | f_w \rangle = \langle v | w \rangle,$$

for any $v, w \in W_e$.

Proof: The first assertion is proved similarly to Theorem V. To prove the second statement, let $E_c = 1, 2, \dots, e$. Define a vector $(f_v^1, f_v^2, \dots, f_v^e)$ for each $f_v \in N_e$, by $f_v^k = f_v(k)$. It is clear that $(f_v^1, f_v^2, \dots, f_v^e)$ is identical to the vector which consists of the first e entries in (v^1, v^2, \dots, v^n) , the 01 vector for v with respect to the standard basis of $W(B)$.

The following calculation establishes the equality of the two inner products:

$$\begin{aligned} \langle f_v | f_w \rangle &= \sum_{k \in E_c} f_v(k) f_w(k) \\ &= \sum_{k \in E_c} f_v^k f_w^k \\ &= \sum_{k \in E_c} v^k w^k \\ &= \langle v | w \rangle. \end{aligned}$$

The last line follows because v and w consist only of external bonds.

We use the notation $N_e(B)$ or $N_{cy}(B)$ if we wish to indicate explicitly the underlying bond graph involved in the chain groups.

Consider $W_{co}(B)$, the co-cycle space of B . This space is also a subspace of W_e . The set of chains, f_w , which correspond to vectors $w \in W_{co}$, is a chain group on E_e . We denote this chain group by $N_{co}(B)$ or N_{co} .

Theorem VII

The chain group N_{co} , when regarded as a vector subspace of $L(E_e)$, is isomorphic to W_{co} .

Proof: This is the dual of Theorem V.

In the sequel, we state theorems for simple bond graphs. For convenience we recall the definition: a *simple* bond graph is one which has an underlying graph with no parallel edges and no isolated junction with one single incident bond (called a *degenerate junction*). Many of the results are true for arbitrary bond graphs but there are often additional technicalities involved in dealing with non-simple bond graphs. Actually, the bond graphs used in modelling physical systems can be restricted to the class of simple bond graphs without any loss.

Theorem VIII

Let B be a simple bond graph. Then, we have

$$N_{cy}^\perp(B) = N_{co}(B) = N_{cy}(B^*) = N_{co}^\perp(B^*).$$

Proof: The equality of $N_{co}(B)$ and $N_{cy}(B^*)$ follows from the definition of $W_{co}(B) = W_{cy}(B^*)$ and application of Theorem V and its dual, Theorem VII. The first equality stated follows from Theorem V of (2) and Theorem VI by using the isomorphism between the subspaces and the chain groups. The last equality is proved similarly.

Definition. We shall denote by $M(B)$ the chain group matroid $M(N_{cy}(B))$. This matroid is called the *cycle matroid* of the bond graph B . Dually, we call the matroid $M(N_{co}(B))$ the *co-cycle matroid* of B and denote it by $M^*(B)$.

Remark 3. Note that the definitions of the cycle and co-cycle matroids of a bond graph B do not use any graph associated with B . In particular the cycle matroid of B is defined even if B is non-graphic. In the case of a graphic bond graph we shall see that there is a close connection between the cycle matroid of the bond graph and the cycle matroid of any graph associated with the bond graph.

Theorem IX

Let B be a simple bond graph. The co-cycle and cycle matroids of B are dual matroids, that is

$$M^*(B) = (M(B))^*.$$

Proof: The cycle matroid of B is $M(B) = M(N_{cy}(B))$. By Theorem III, its dual

matroid is $M(N_{cv}^\perp)$. But this is $M(N_{co})$, by Theorem VIII. By definition this is $M^*(B)$.

The result of Theorem IX justifies the notation $M^*(B)$ for the co-cycle matroid of a bond graph. We now establish a connection between the matroids of a bond graph and those of its dual bond graph.

Theorem X

Let B be a simple bond graph. The co-cycle matroid of B is the cycle matroid of B^* , the dual bond graph of B :

$$M^*(B) = M(B^*).$$

Proof: The cycle matroid of B^* is $M(B^*) = M(N_{cv}(B^*))$. But this matroid is $M(N_{cv}^\perp(B))$, by Theorem VIII. By Theorem III this is the dual matroid of $M(N_{cv}(B)) = M(B)$. This dual is $M^*(B)$, by Theorem VIII.

IV. Equivalence of Bond Graphs

Definition. Let B be a bond graph with cycle matroid $M = M(B)$ and co-cycle matroid M^* . We denote by $\mathcal{B}(M)$ the class of all bond graphs whose cycle matroid is isomorphic to M . $\mathcal{B}^*(M)$ will denote the class of all bond graphs whose co-cycle matroid is isomorphic to M^* .

Definition. If B_1 and B_2 have isomorphic cycle matroids we say that B_1 and B_2 are *cycle equivalent*. Dually B_1 and B_2 are *co-cycle equivalent* if they have isomorphic co-cycle matroids.

Theorem XI

Simple bond graphs B_1 and B_2 are cycle equivalent if and only if they are co-cycle equivalent.

Proof: The result follows immediately from Theorem X, for two matroids are isomorphic if and only if their dual matroids are isomorphic.

Corollary. For simple bond graphs we have

$$\mathcal{B}(M) = \mathcal{B}^*(M^*).$$

Definition. We shall call two bond graphs B_1 and B_2 *equivalent* and write $B_1 \approx B_2$ if B_1 and B_2 are cycle (or equivalently, by Theorem XI co-cycle) equivalent.

Remark 4. It is well-known that, even after choosing a definite set of physical components for a bond graph model of a physical system, there exist many equivalent (and apparently quite different) system bond graphs for the same model. We have now made quite precise the exact nature of this equivalence. We have established an infinite class of equivalent bond graphs, any one of which could be used as a junction structure for the system bond graph. In particular, the bond graphs in $\mathcal{B}(M)$ are all equivalent and all have the same cycle and co-cycle matroid. Hence, any one of these bond graphs can represent the combinatorial structure of the system.

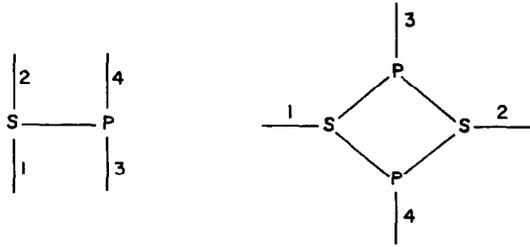


FIG. 1. Equivalent bond graphs.

We can express the result of Theorem XI in the vector space terminology of (1, 2) and obtain an equivalent theorem.

Theorem XII

Simple bond graphs B_1 and B_2 are s -equivalent if and only if they are p -equivalent. If this is the case then B_1 and B_2 are equivalent in the matroid sense.

Proof: Now B_1 and B_2 are s -equivalent if and only if B_1 and B_2 are cycle equivalent. Similarly B_1 and B_2 are p -equivalent if and only if B_1 and B_2 are co-cycle equivalent. The result follows from Theorem XI.

Corollary. A simple bond graph B is co-graphic if and only if $W_{co}(B)$ is isomorphic to the co-cycle space of some graph.

The result of Theorem XII was stated in (2), and the proof was deferred to this paper, since it is a trivial application of the bond graph matroid theory.

Example 1. Consider the two bond graphs illustrated in Fig. 1. The cycle spaces of these were shown to be isomorphic, in (1). Thus, by Theorem XII these bond graphs are equivalent. This result is known in the bond graph literature as the *diamond equality* (see 16) although, of course, this is not stated using the precise definition given above for the equivalence of bond graphs.

The equivalence of the two bond graphs of Example 1 is important, since it is fundamental to the process of transforming a bond graph into an equivalent one. this procedure, for example, might be applied in the simplification of a bond graph after constructing a model using the systematic method. It can be applied when some, or all, of the incident bonds are internal bonds. We prefer to call the bond graphs of Fig. 1 *coupled* and *de-coupled* equivalent bond graphs, the 2-junction bond graph being the de-coupled one.

Example 2. Consider the bond graphs shown in Fig. 2. Each one of these can be obtained from one of the others by permuting the bonds on pairs of opposite junctions on the ring. Suppose we keep the junctions fixed and consider the sequence of bond numbers on the junctions. Then B_2 can be obtained from B_1 by the permutations (1 2) and (4 5), where ($j k$) means that the bonds j and k are permuted in the sequence. Similarly B_3 corresponds to the permutations (2 3) and (5 6); B_4 corresponds to (3 4) and (6 1). It is clear that there are precisely four topologically distinct bond graphs which can be obtained using these permutations. Now it is quite simple to check that these are all equivalent bond graphs, since the cycle matroid of each has the following class of circuits :

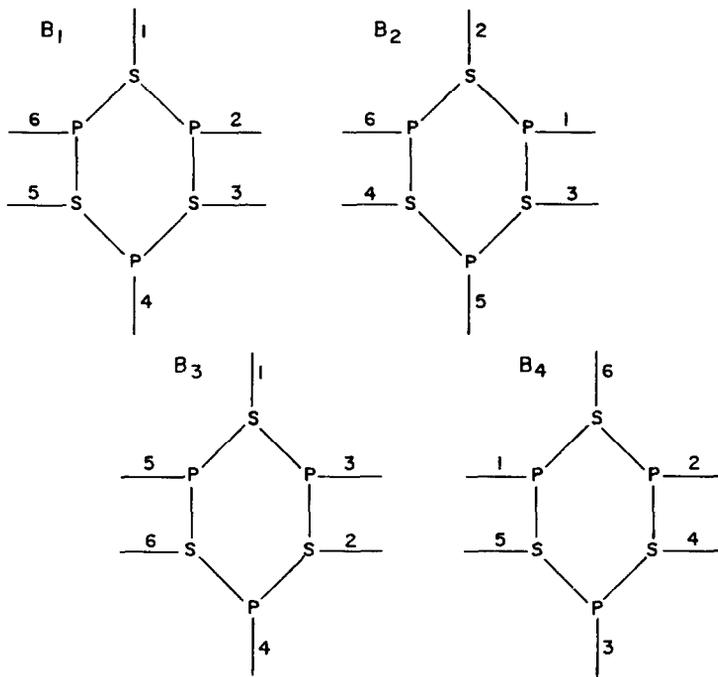


FIG. 2. Equivalent ring bond graphs.

$$\{612, 234, 456, 135, 1346, 1245, 2356\}.$$

For B_1 , the generating circuits, obtained from elementary junction vectors, are $\{126, 234, 456\}$. The circuit 135 is the sum of these three circuits:

$$135 = 612 + 234 + 456.$$

In the case of the other bond graphs the circuits of the cycle matroid are generated by three circuits, each with three bonds, obtained from s -type vectors on the three s -junctions. The fourth circuit with three bonds is the sum of these three generating circuits and consists of the bonds on the three s -junctions.

Continuing with the example, consider the bond graph obtained by coupling two of the neighbouring junctions of B_1 , say the sub-bond graph illustrated in Fig. 3(a). The coupled equivalent bond graph for this sub-bond graph is shown in Fig. 3(b) and, after re-connecting the internal bonds and contracting internal bonds joining similarly labelled junctions, we arrive at the equivalent bond graph shown in Fig. 3(c). Examining this bond graph, we can see that the circuits of the cycle matroid are generated by

$$\{612, 234, 2356\}.$$

These generate the same class of circuits given above and therefore we see explicitly the equivalence of this bond graph to the others.

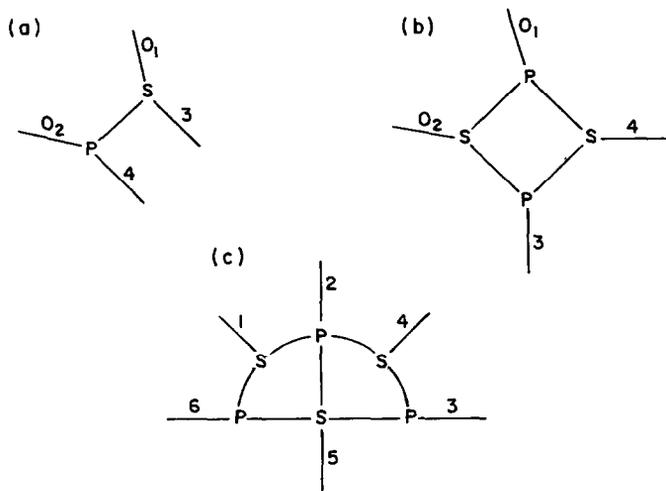


FIG. 3. Equivalent bond graph obtained by coupling. (a) Sub-bond graph for coupling; (b) coupled equivalent sub-bond graph; (c) re-connected equivalent bond graph.

V. Bond Graphs and Linear Graphs

We now turn to the problem of associating a graph with a bond graph and the inverse problem. The following definition is clearly equivalent to the one used in the earlier papers of the series. It is simply a matroid re-statement of that definition.

Definition. A bond graph B is *graphic* if there is a graph G such that $M(B)$ is isomorphic to $M(G)$. If G is any such graph then G and the bond graph B are said to be *associated*. If B and G are associated we write either $B = B(G)$ or $G = G(B)$ to denote this relationship. Dually, if there exists a graph G such that $M(B)$ is isomorphic to $M^*(G)$, we say that B is *co-graphic* and G and B are *co-associated*.

Definition. We denote by $\mathcal{G}(M)$ the class of graphs G for which $M(G) \approx M$. Similarly $\mathcal{G}^*(M)$ will denote the class of graphs G such that $M^*(G) \approx M$, or, equivalently, $M(G) \approx M^*$.

Remark 5. The definitions and results above establish a one to one correspondence between $\mathcal{G}(M)$ and $\mathcal{B}(M)$, that is, between equivalence classes of graphs and equivalence classes of bond graphs. This correspondence establishes exactly when a system graph model of a physical system is topologically equivalent to a system bond graph model. This will occur precisely when the graph is in $\mathcal{G}(M)$ and the bond graph is in $\mathcal{B}(M)$. In addition to establishing when two models are topologically equivalent these results demonstrate the topological equivalence of the two modelling techniques, one of the motivations for our study of the mathematical structure of bond graphs.

We continue with a discussion of duality as related to graphs, bond graphs and their matroids.

Definition. Let G be a graph and B a bond graph. In (1), we defined $\mathcal{B}(G)$. In the matroid terminology the equivalent definition and its dual are:

$$\mathcal{B}(G) = \mathcal{B}(M(G)) \quad \text{and} \quad \mathcal{B}^*(G) = \mathcal{B}^*(M(G)).$$

Theorem XIII

A bond graph B is graphic if and only if there exists a graph G such that $M^*(B) \approx M^*(G)$. Dually, B is co-graphic if and only if there is a graph G such that $M^*(B) \approx M(G)$.

Proof: The theorem follows immediately from the previous results.

Theorem XIV

Let G be a graph and G^* be any (abstract or geometric) dual of G . Then a bond graph $B \in \mathcal{B}(G)$ if and only if $B^* \in \mathcal{B}(G^*)$. Equivalently we have

$$\mathcal{B}(G^*) = \mathcal{B}^*(G).$$

Proof: We have

$$\mathcal{B}(G^*) = \mathcal{B}(M(G^*)) = \mathcal{B}(M^*(G)) = \mathcal{B}^*(M(G)) = \mathcal{B}^*(G).$$

Note that the statement $M^*(G) \approx M(G^*)$ is a standard result in matroid theory (see Section III).

Definition. Let B be a bond graph. We make the following definitions of classes of associated and co-associated graphs:

$$\mathcal{G}(B) = \mathcal{G}(M(B)) \quad \text{and} \quad \mathcal{G}^*(B) = \mathcal{G}(M^*(B)).$$

The justification for the notation $\mathcal{G}^*(B)$ comes from the following theorem.

Theorem XV

Let G be a graph and let B be in $\mathcal{B}(G)$. Then G has a dual graph if and only if B^* is graphic, or, equivalently, B is co-graphic. Any dual G^* is in $\mathcal{G}^*(B)$. That is, $G \in \mathcal{G}(B)$ if and only if any dual $G^* \in \mathcal{G}^*(B)$.

Proof: The dual bond graph B^* is graphic if and only if there exists a \hat{G} in $\mathcal{G}(B^*)$. By definition we would have, in this case, that $M(B^*) = M(\hat{G})$. But $M(B^*) = M^*(B)$ by Theorem X and this is $M^*(G)$ by Theorem XIV. Thus we see that B^* is graphic if and only if there exists a graph \hat{G} such that $M^*(G) = M(\hat{G})$. It is a standard result in matroid theory that \hat{G} is a dual of G (see Section II.1). The assertions about $\mathcal{G}^*(B)$ then follow from the definition of this class of graphs.

Corollary. Let G be a graph with dual G^* . Then $G \in \mathcal{G}(B)$ if and only if $G^* \in \mathcal{G}^*(B)$.

Proof: We have $G \in \mathcal{G}(B)$ if and only if $G^* \in \mathcal{G}^*(B)$. But also, we have

$$\mathcal{G}^*(B) = \mathcal{G}(M^*(B)) = \mathcal{G}(M(B^*)) = \mathcal{G}(B^*).$$

Remark 6. We can see that the result of the corollary above is equivalent to the statement that graph G is associated with a bond graph B if and only if any dual graph G^* is associated with the dual bond graph B^* . This result was mentioned in (2) but we noted there that the proof would be deferred until we had established the theory of bond graph matroids.

Remark 7. Theorem XV establishes when a graph G fails to have a dual graph. This occurs precisely when any bond graph associated with G fails to be co-graphic.

Hence, we have shown that a graph G fails to have a dual graph precisely when $M(G)$, the cycle matroid of G , fails to be co-graphic. This establishes a well-known result in matroid theory.

Remark 8. A bond graph gives a pictorial representation of a certain class of matroids, namely the ones which are the matroids of chain groups over $GF(2)$, or, equivalently, by Theorem II, the ones which are representable over $GF(2)$. Such matroids are called *binary* matroids. Thus a bond graph can be regarded as a pictorial representation of a binary matroid. We discuss the matrix representability of bond graph matroids in the next paper in this series. In physical system modelling we can apply a matrix representation to find fundamental sets of equations for either a system bond graph or system graph model.

Remark 9. In (1) we commented that it is a very difficult problem to determine whether a bond graph is graphic. We can see now that this problem is equivalent to deciding whether a given binary matroid is graphic, which was solved by Tutte (13). Tutte's theorem is regarded by some authors as being one of the deepest results in combinatorics. It provides a complete matroid generalization of the famous Kuratowski theorem concerning planarity of graphs. For a simple exposition of the statement of Tutte's theorem and a general discussion of these results see (11).

VI. Examples

In this section we give detailed examples of some of the concepts which were considered above. We also describe a systematic method of obtaining generating circuits for the class of dependent sets of the cycle matroid of a bond graph, from which it is simple to extract the class of circuits.

Example 3. Consider the graph G with planar representation shown in Fig. 4(a). The cycle space, $W_{cy}(G)$ is generated by the cycles 14, 25 and 123. Thus $M(G)$ has circuits $\{14, 25, 123, 234, 135, 345\}$. Now consider the bond graph B shown in Fig. 4(b). The cycle space of B , $W_{cy}(B)$, has basis $\{14, 25, 123\}$. Hence $N_{cy}(B)$ is the chain group generated by the chains f_1, f_2 and f_3 , which are listed below:

	1	2	3	4	5
f_1	1	0	0	1	0
f_2	0	1	0	0	1
f_3	1	1	1	0	0
f_1+f_2	1	1	0	1	1
f_1+f_3	0	1	1	1	0
f_2+f_3	1	0	1	0	1
$f_1+f_2+f_3$	0	0	1	1	1

The numbers above the columns refer to the bonds of B . Each of the chains in the list is an elementary chain in N_{cy} , with the exception of f_1+f_2 . Hence the circuits of $M(B)$, the cycle matroid of B , are $\{14, 25, 123, 234, 135, 345\}$. Plainly

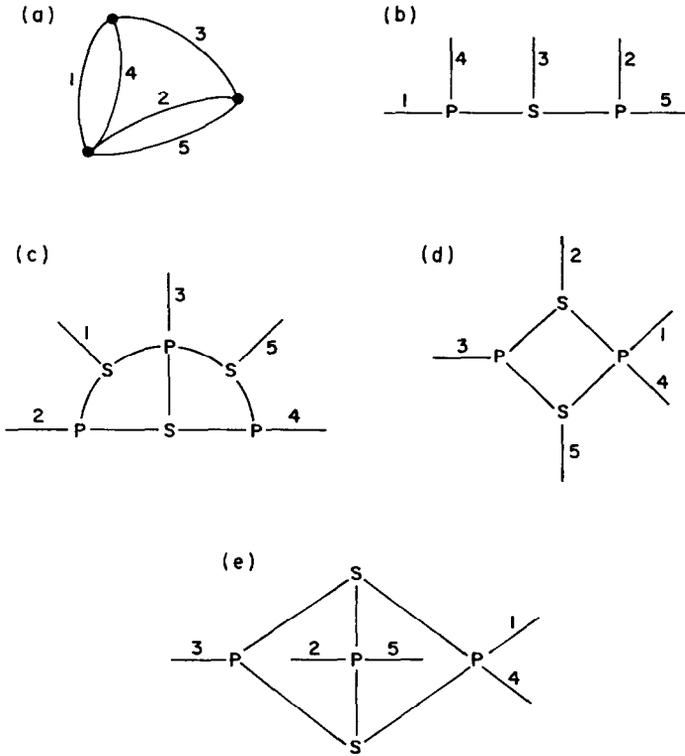


FIG. 4. Graph and equivalent associated bond graphs.

$M(B) \approx M(G)$ and so we have $G \in \mathcal{G}(B)$ and $B \in \mathcal{B}(G)$. The dependent sets of $M(B)$ are the circuits listed above, together with the set 1245. We see that this dependent set corresponds to the union of disjoint cycles 14 and 25 of G . In Figure 4(c, d, e) we show three quite different bond graphs each of which is also associated with the graph G . The reader should readily be able to check that these are equivalent bond graphs in $\mathcal{B}(G)$, by finding their cycle matroids.

In practice, it is not necessary to write down explicitly all the chains in N_{cp} , in order to find the dependent sets, and hence the circuits, of $M(B)$. We can write down a list of dependent sets in $M(B)$ following the simple procedure below, which uses the bond graph directly:

- (1) Write the s -type elementary junction vector, s , for each s -junction of B . If the s -junction is not an isolated junction of B , this vector will contain at least one internal bond. Now eliminate the internal bond(s) in s by choosing an appropriate linear combination of s , together with p -type vectors on adjacent p -junctions. If we introduce new internal bonds, these are eliminated by adding the appropriate elementary junction vectors. Since the bond graph has a finite number of bonds and junctions each of the paths from the original s -junction must eventually reach an external p -junction. Hence we can eliminate all the internal bonds and obtain a vector in the cycle space of B .

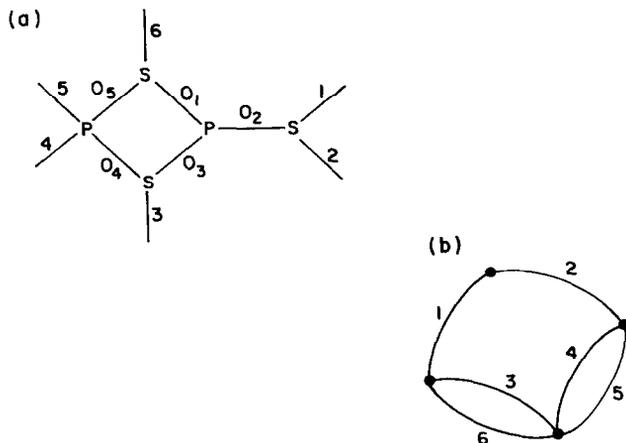


FIG. 5. Bond graph and an associated graph.

- (2) For the p -junctions of B which have an external degree of 2 or more, we write one p -type elementary junction vector containing each external bond on that p -junction and no internal bond.

The list of sets obtained above is a spanning set for $W_{cy}(B)$. If B is not a tree (or forest) bond graph, the list above may contain redundancies which can be discarded. Also, some care is needed to ensure that all independent paths are included in the procedure, in order that no vectors are omitted. We can generate a list of dependent sets of the cycle matroid of B , $M(B)$, by symmetric differences. The dependent sets which correspond to elementary chains are the circuits of $M(B)$.

When applying the procedure described above, we must find, for each s -junction of B , a vector which is s -equivalent to the s -type elementary junction vector on that junction, but has only external bonds. Thus we do not require the specific linear combination which eliminates the internal bonds. For calculations, then, we simply write a sequence of s -equivalent vectors, beginning with the junction vector and ending with the required vector. For instance, in the example below, " $6o_5o_1 \stackrel{s}{=} 64o_1$ " means the two vectors are s -equivalent and records the result of the combination $6o_5o_1 + 4o_3 = 64o_1$.

Example 4. Consider the bond graph shown in Fig. 5(a). The following calculations show the vectors in (1) above:

$$\begin{aligned}
 6o_5o_1 &\stackrel{s}{=} 64o_1 \stackrel{s}{=} 64o_2 \stackrel{s}{=} 6412 \\
 12o_2 &\stackrel{s}{=} 12o_3 \stackrel{s}{=} 123o_4 \stackrel{s}{=} 1234 \\
 (3o_3o_4 &\stackrel{s}{=} 34o_3 \stackrel{s}{=} 34o_2 \stackrel{s}{=} 3412).
 \end{aligned}$$

Finally we write down a p -type vector for the p -junction with 2 incident external bonds: 45.

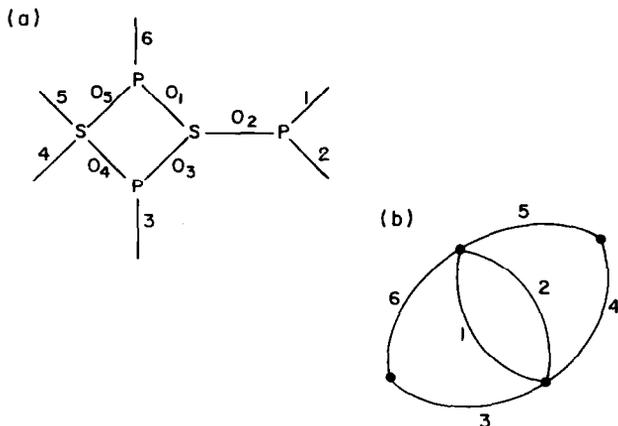


FIG. 6. Bond graph and an associated graph.

Note that one of the junctions in (1) resulted in a redundant vector in the list. The dependent sets of $M(B)$ are thus generated by $\{1246, 1234, 45\}$. We can now write down the class of dependent sets of $M(B)$:

$$\{1234, 1246, 45, 36, 1235, 1256, 3456\}.$$

The only dependent set which is not a circuit is 3456, which is the union of the circuits 45 and 36.

Using the inverse cut and paste method we can construct a graph $G(B)$ associated with B . This is shown in Fig. 5(b). Each of the dependent sets of $M(B)$, with the exception of 3456, is a cycle of the graph $G(B)$. The set 3456 is the disjoint union of cycles 36 and 45.

Example 5. The bond graph shown in Fig. 6(a) is the dual of the bond graph used in Example 4. The dependent sets which generate those of $M^*(B)$ are found to be:

$$45o_4o_5 \stackrel{s}{=} 4536$$

$$o_1o_2o_3 \stackrel{s}{=} 631$$

$$12.$$

Thus for $M^*(B) = M(B^*)$ the class of dependent sets is

$$\{12, 136, 3456, 236, 123456, 145, 245\}.$$

These, with the exception of 123456, are the circuits of $M(B^*)$.

We can use the inverse cut and paste method, applied to B^* , to construct a graph associated with B^* . This is shown in Fig. 6(b). Clearly this $G(B^*)$ is a dual graph of $G(B)$. $G(B^*)$ is co-associated with B and $G(B)$ is co-associated with B^* . The dependent sets of $M^*(B)$, are co-cycles or unions of disjoint co-cycles of the graph $G(B)$. Dually, the dependent sets of $M^*(B) = M(B^*)$ are also cycles and unions of disjoint cycles of the graph $G(B^*)$.

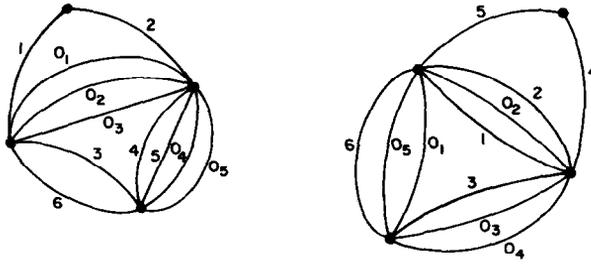


FIG. 7. Augmented graphs.

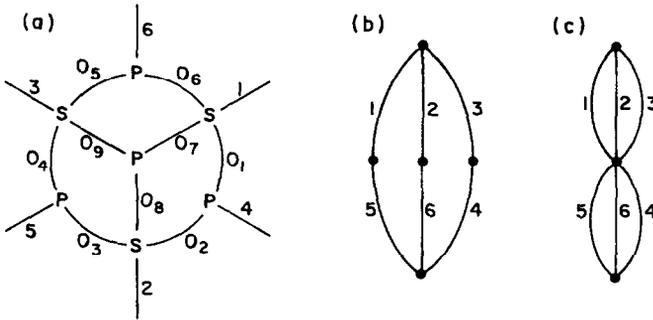


FIG. 8. Failure of the inverse cut and paste method.

For completeness we show, in Fig. 7, the augmented graphs obtained from B and B^* , when applying the inverse cut and paste method to the bond graphs. Note that the two augmented graphs are *not* duals. The internal edges do not correspond, in any way, between the two augmented graphs.

Example 6. Consider the bond graph, B , shown in Fig. 8(a). The s -space of B , W_s , has dimension 11. Bases for the cycle, internal and coupling spaces of B are given below :

$$W_{cy} : \{1256, 1345\}$$

$$W_i : \{0_70_8, 0_80_9, 0_10_2, 0_30_4, 0_50_6\}$$

$$W_c : \{40_1, 50_3, 60_5, 10_10_60_7\}.$$

If we attempt to apply the inverse cut and paste method to B , we discover that it is impossible to construct an augmented graph, G^a , with precisely the cycles given by the s -space $W_s(B)$. All the required cycles can be accommodated using a graph like G_1 , shown in Fig. 8(b), except for the cycle $10_10_60_7$, which requires the three vertices to be joined into one vertex, as in the graph G_2 of Fig. 8(c). However, G_2 has a 4-dimensional cycle space with basis $\{12, 13, 45, 46\}$ and hence this cannot be a graph associated with B .

The inverse cut and paste method fails to produce any graph associated with B . However the graph G_1 plainly has the required cycle space and is associated with B . This provides the example referred to (I) in comment (v) concerning the inverse

cut and paste method. Even though B is graphic and has a planar associated graph, G_1 , the method fails.

We can explain this failure using matroid theory. If it is possible to construct an augmented graph using the method then the cycle matroid of this augmented graph must be graphic. In the case of the bond graph above, this matroid has its dependent sets generated by the 11 sets listed above under the decomposition of W_s . This matroid is non-graphic and so we cannot construct G^a .

VII. On Physical System Modelling

In physical system modelling the combinatorial information which is recorded in either a bond graph model or an equivalent linear graph model is the topology of interconnection of the components. This cycle (and co-cycle) structure can be expressed concisely by using a matroid and its dual. For a linear graph model, the information is recorded visually. However, for a bond graph model the information is encoded in the symbols used for the junctions. The bond graph picture is an explicit record of the matroid structure, whereas the graph is an intermediate visual aid which records the information implicitly.

A mathematical procedure which produces system equations using a combinatorial procedure is called a *formulation method*. Any such formulation uses the cycle and co-cycle structure of the system, since it is on these that the generalized Kirchhoff laws apply. Whether we use a graph or a bond graph as a pictorial diagram of the combinatorial structure is irrelevant to the procedure. In either the graph or bond graph model it is a matrix representation of the cycle matroid of the system structure which is used. The pictorial device is used as an intermediate to obtaining this matrix. We investigate matrix representations of bond graph matroids in (8).

The physical interpretation given to the through/across variables of the graph edges, or, equivalently the effort/flow variables of the bonds, is called the *physical analogy*. This choice influences the choice of the combinatorial model used for the system structure. Reversing the choice of variables is called *dualizing* the analogy. In Remark 5 of (2) we noted that dualizing the analogy used in a model requires a dualizing of the combinatorial structure of the model. For linear graph models the choice is between a graph or its dual graph. With a bond graph model the choice is between a bond graph or its dual bond graph. Of course dualizing the physical analogy will also involve appropriate changes to some of the physical components.

In the context of matroid theory we can see that this procedure changes the combinatorial model from a matroid to its dual matroid. In the case of a bond graph model the dual structure always exists, since every bond graph has precisely one dual bond graph. However not every graph has a dual graph (in particular, a non-planar graph has no dual). Thus it is not always possible to dualize the analogy when using a linear graph model. This must not be seen as an inherent failure of the modelling method. Rather it indicates that the pictorial device used as a diagram of the combinatorial structure has a limitation which the bond graph picture does not have. In (17) there is a discussion of analogy and dualogy. By considering

matroids we can see that the choice of analogy is of no significance, for the dual structure is always available.

VIII. Conclusions and Summary

By using chain group matroids derived from the cycle and co-cycle vector spaces of a bond graph, we defined the cycle and co-cycle matroids of a bond graph. The relationship between these structures was investigated and various results were proved. Duality theory was seen to be very clear in the context of bond graph matroids.

A precise definition of equivalent bond graphs was given: these have the same cycle (and co-cycle) matroid. This establishes an equivalence class of bond graphs, any one of which could be used as a pictorial diagram of the combinatorial structure of the matroid. This explains why a system bond graph model can exist in many equivalent and quite distinct forms and provides a method for determining equivalence.

The connections between bond graphs and linear graphs were investigated. These are different pictorial diagrams of the same combinatorial structure. We defined an equivalence class of graphs, those with the same cycle (and co-cycle) matroid. Thus, by using matroids, a one to one correspondence was provided between equivalence classes of graphic bond graphs and equivalence classes of graphs. Results were proved connecting the duality theory of bond graphs. Results were proved connecting the duality theory of bond graphs, linear graphs and matroids.

Several examples were given to illustrate the theory. We concluded with a discussion of the significance of some of the results with respect to physical system modelling.

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