The Mathematical Foundations of Bond Graphs—II. Duality

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ABSTRACT: *A discussion of duality continues the development of a mathematical theory for bond graphs as combinatorial structures. The dual of a bond graph is defined and its associated vector spaces are the co-spaces of the original bond graph. The relationship between the algebraic structures and their duals is explored. The theory is used to justify the construction of a simplified bond graph, the proper contraction, which is equivalent to the original bond graph. An inner product is defined and an orthogonality theorem between primal and dual vector spaces of a bond graph is proved.*

I. Introduction

We continue the analysis of combinatorial bond graphs begun by the authors, see (**1**), where elementary mathematical concepts from linear algebra were used to develop an independent theory for non-directed bond graphs. In this paper, we expand the theory to include the concept of duality, which appears to be extra-ordinarily natural in the context of bond graphs. The terminology, notation and special symbols which we defined in (**1**) are used freely here. We refer the reader to the earlier paper for the definitions.

The authors' original motivation for analysing bond graphs was to investigate the connections between two apparently distinct techniques used to model physical systems—the system graph and bond graph models. System graphs are formed when a physical system is modelled using a linear graph to indicate the inter-connection topology of the components (**2**). Linear graphs are combinatorial structures with a well-known mathematical theory (**3**).

The second method uses bond graphs, or more precisely, system bond graphs, in which the combinatorial information is encoded using symbols (**4**). There is no existing theory which considers a bond graph as a mathematical structure, independent of any application to physical system modelling. In this series of papers on the mathematical foundations of bond graphs, we hope to provide such a theory in which we analyse the properties and capabilities of bond graph notation as a pictorial representation of combinatorial information.

In (**1**), we gave a precise definition of a combinatorial bond graph, B. We then coded combinatorial information by defining a vector space structure on B. This algebraic structure is used implicitly in every system bond graph model. The theory was developed and various results proved, many of which have significance in the application to physical system modelling.

In Part III (**5**) we study the combinatorial properties of bond graphs using matroids, to which they are intimately connected. A detailed discussion of causality in Part IV (**6**) illustrates one of the more powerful aspects of the bond graph concept. Once we have established a theory for non-directed bond graphs, we can investigate the theory of orientation for directed bond graphs (i.e. including power half-arrows), which is essential for physical system modelling. Part V in the series (**7**) develops this theory.

Armed with a rigorous theory on which to base system bond graph modelling, we are in a position to compare the system graph and system bond graph modelling techniques. When viewed on this level, the mathematical equivalence of the two methods becomes quite transparent. By analysing this equivalence precisely, a complete unification of the two modelling techniques should be possible. Having demonstrated this, we hope that practitioners of one or the other technique will become less isolated from each other. The historical division which exists has no practical justification, for bond graphs and linear graphs are simply different pictorial diagrams of the same combinatorial information.

There are many possibilities for cross-fertilization of the two modelling techniques. Any formulation technique which is used in one context may equally well be used in the other, although there may be technical problems associated with the translation. An example of such an application is given in Birkett, Roe and Thoma (**8**), where we present a traditional network technique in a bond graph context.

A particular benefit of this theory is in the area of computer software for system modelling. For example, existing simulation software expressed in terms of system bond graphs may be translated into software applicable to a linear graph model and *vice versa*. Using the theory we develop in these papers, the translation should be quite simple.

From an analytical point of view, many of the problems associated with the bond graph notation are mathematical, for instance, problems with assignment of causality and orientation, see Perelson (**9**, **10**). Such problems are essentially combinatorial problems and so it is natural to discuss them in a combinatorial setting. We also discuss the results of Ort and Martens (**11**) concerning junction structure matrices. These matrices are concrete representations of our algebraic structures. Thus, results concerning these matrices follow quite simply from the theorems in (**5**, **6**) and Section VI of this paper.

Several authors have given examples of bond graphs which cannot represent a physical system. For instance, Perelson (**10**) presents such a bond graph which he calls a "non-realizable bond graph." We show in (**7**) that this bond graph cannot represent a physical system due to a failure of the half-arrows to provide an orientation of the combinatorial structure of the system. Other examples of such "non-physical" bond graphs are known, but in all cases the problem is caused by an orientation and/or causality problem.

In (**1**) we showed that there is a class of bond graphs which cannot represent a physical system for basic combinatorial reasons. We also provided an example of such a bond graph, called a *non-graphic* bond graph. These comments refer to non-directed bond graphs and, of course, since we have not yet introduced the concept of causality, this cannot enter into the discussion. The non-physical character of such bond graphs is entirely combinatorial in nature. Even though this lies at the heart of the bond graph notation, it is a point which has been overlooked in the literature. We complete the discussion of this point in this paper.

For convenience, we used some elementary concepts from graph theory in developing the theory in (**1**), although, as we noted in that paper, the theory of combinatorial bond graphs can exist without any reference to graph theory. In addition, wherever possible we have established connections between bond graphs and linear graphs, since these results are very significant in the comparison of the two modelling methods. Therefore, in (**1**), we included an appendix which gives definitions of all the graph theory terminology. We use the same definitions in this paper, and refer the reader to (**1**) for the summary. As well as those definitions, in this paper we must use some concepts concerning duality for graphs. Therefore, we include an appendix here which summarizes graph duality.

We begin by defining the dual bond graph and the associated vector space, called the *p-space*, in Section II. The co-spaces of a bond graph are discussed in Section III and s-equivalence of vectors in Section IV. We define the proper contraction of a bond graph in Section V, proving a theorem which justifies a common procedure used in the bond graph literature. In Section VI we introduce an inner product and establish an important result concerning orthogonality for bond graph vector spaces. This is a result which is known in the context of junction structure matrices (**11**) and also for cutset and circuit matrices of graphs (**12**). Both these results are, in fact, corollaries of our Theorem V. We conclude with an example in Section VII and a summary of the main results in Section VIII.

II. The Dual Bond Graph

Definition. Let B be a bond graph. The *dual bond graph* of B, denoted by B^*, is the bond graph which is identical to B, except that the labels on the junctions are exactly opposite to those of B.

By definition, the p-junctions of B correspond to the s-junctions of B^* and *vice versa.* We shall see that the relation between a bond graph and its dual is similar to that between a graph and one of its dual graphs (the term "dual graph" will be used for either a geometric or abstract dual). In the case of a bond graph, though, precisely one dual bond graph, B^*, exists for each B. Corresponding to the situation of a graph which has no dual graph, we shall see that some bond graphs may be graphic while their dual is non-graphic.

Each of the structures defined on a bond graph, B, has a dual structure which has the same form, but uses the dual bond graph, B^*. In this paper we investigate these dual structures and also the relationship between the structures and their duals.

We begin by considering the bond space and the s-space of a bond graph. The internal and external bonds of B are identical to those of B^*, hence the bond space, $W(B)$, constructed in (**1**), is identical to the bond space, $W(B^*)$, of the dual bond graph. However, the definition of the s-space of B uses the junction labels asymmetrically, thus the s-space of B^* will be quite different, in general, from that of B. We make the following definition.

Definition. The *p-space*, $W_p(B)$, *of a bond graph B* is defined by

$$W_p(B) = W_s(B^*).$$

The s-space of B is defined in (**1**) as the linear span of a certain collection of vectors, called the elementary junction vectors of B. An s-type vector consists of all the bonds incident on one particular s-junction and a p-type vector consists of precisely two of the bonds incident on a particular p-junction. We can use the duals of these elementary vectors, which are defined below, to generate the p-space of B.

Definition. Each s-type elementary junction vector of B^* will be called a *p^*-type elementary junction vector* of B. Similarly, a p-type elementary junction vector of B^* will be called an *s^*-type elementary junction vector* of B. The s^*- and p^*-type elementary junction vectors of a bond graph, B, will be called collectively, the *dual elementary junction vectors* of B.

We can use the s^*- and p^*-elementary junction vectors of a bond graph, B, to give an equivalent definition of the p-space of B. The p-space of B, $W_p(B)$, is the linear span of all the dual elementary junction vectors of B.

Lemma 1. Let B be a bond graph. Any vector in $W_s(B)$ can be written as a linear combination of s- and p-type elementary junction vectors, in which each vector occurs without repetition. Dually, any vector in $W_p(B)$ can be written as a linear combination of p^*- and s^*-type dual elementary junction vectors without repetition.

Proof: By definition, any vector in $W_s(B)$ can be decomposed as a linear combination of s- and p-type vectors. For any elementary junction vector which occurs in this linear combination an even number of times, there is no change to the sum if all entries are deleted. Also, for a vector which occurs an odd number of times, all occurrences but one can be deleted without changing the sum of the combination. The statement for $W_p(B)$ follows by considering the dual bond graph, B^*.

Definition. The linear combinations in Lemma 1, without repetitions of junction vectors, will be called *elementary decompositions* of vectors in W_s and W_p.

The linear combinations in Lemma 1 are not unique since the elementary junction vectors are not linearly independent, as the following example shows.

Example 1. In Fig. 1(a), we show a bond graph which has the following elementary junction vectors:

$$s: 12o_1, 456o_2$$

$$p: 3o_1, 3o_2, o_1o_2$$

$$p^*: 3o_1o_2$$

$$s^*: 1o_1, 12, 2o_1, 4o_2, 5o_2, 6o_2, 45, 46, 56.$$

Either of the following linear combinations of elementary junction vectors is an elementary decomposition of the vector $v = 12456 \in W_{cy}(B)$:

$$v = 12o_1 + o_13 + 3o_2 + o_2456$$

$$= 12o_1 + o_1o_2 + o_2456.$$

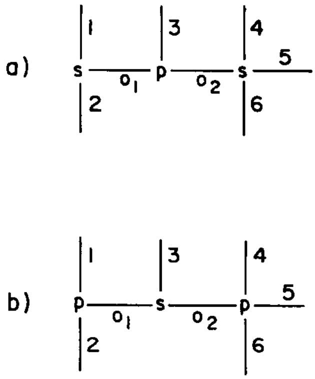

FIG. 1. Bond graph of Example 1. (a) Bond graph B; (b) dual bond graph B^*.

The p-space, $W_p(B)$, is the subspace spanned by the p^*- and s^*-type vectors listed above. We can select a basis for the p-space from these, by choosing a maximal linearly independent subset, for instance $\{3o_1o_2, 12, 1o_1, 4o_2, 45, 46\}$. The p-space of B is seen to be 6-dimensional.

For comparison, one choice of a basis for the s-space of B would be $\{12o_1, 3o_1, 3o_2, 456o_2\}$. The s-space is seen to be 4-dimensional.

To illustrate the definition of dual bond graph, the dual bond graph, B^*, is shown in Fig. 1(b).

We now prove a useful combinatorial lemma, which relates elementary junction vectors and their duals.

Lemma 2. Let B be a proper simple bond graph. An elementary junction vector can have no bond in common with a dual elementary junction vector except for the following cases:

(i) An s-type and an s^*-type both formed on the same s-junction have exactly two common bonds. Dually, a p-type and a p^*-type formed on the same p-junction have exactly two common bonds.

(ii) An s-type and a p^*-type formed on adjacent junctions have exactly one common bond.

(iii) A p-type and an s^*-type formed on adjacent junctions may have either one common bond or no common bonds. The common bond is the internal bond incident on both junctions, provided this is contained in both of the junction vectors.

Proof: First, we note that s- and s^*-type vectors formed on the same s-junction have in common precisely the two bonds of the s^*-type vector. Dually, p- and p^*-type vectors on the same p-junction have in common the two bonds of the p-type vector.

The bond graph is proper and so distinct junctions with the same label will be non-adjacent. For such junctions, and also for non-adjacent junctions with different labels, there can be no common bonds between elementary and dual elementary junction vectors.

The s-type and p^*-type vectors include all bonds on the relevant junctions. If these junctions are adjacent there will be exactly one bond in common, the one internal bond incident on both of the junctions.

Similarly, p-type and s^*-type vectors formed on adjacent junctions have in common the internal bond which joins the junctions, provided this bond is contained in each of the two vectors.

Remark 1. For statement (ii) in Lemma 2, the dual statement is identical, since s-type and p^*-type vectors of B correspond, respectively, to p^*-type and s-type vectors of B^*. Similarly, p-type and s^*-type vectors of B correspond, respectively, to s^*-type and p-type vectors of B^*. Hence the dual of statement (iii) is the same statement.

Notation. The following notation will be convenient. We shall denote the s-type elementary junction vector formed on s-junction s, by the same letter, s. The p^*-type vector formed on p-junction p, will be denoted by p^*. It should be quite clear from the context whether the letter denotes the junction or the vector.

III. Co-spaces of a Bond Graph

Definition. Let B be a bond graph. The *co-cycle space* of B is the cycle space of B^*. The *co-internal space* of B is the internal space of B^*. Any coupling space of B^* is called a *co-coupling* space of B. We use the following notation for these spaces:

$$W_{co}(B) = W_{cy}(B^*)$$
$$W_{ci}(B) = W_i(B^*)$$
$$W_{cc}(B) = W_c(B^*),$$

where $W_c(B^*)$ is any of the coupling spaces of B^*.

The p-space of a bond graph has direct sum decompositions, dual to the decompositions of the s-space of the dual bond graph. These are given by

$$\begin{aligned} W_p(B) &= W_s(B^*) \\ &= W_{cy}(B^*) \oplus W_i(B^*) \oplus W_c(B^*) \\ &= W_{co}(B) \oplus W_{ci}(B) \oplus W_{cc}(B). \end{aligned}$$

Definition. Two bond graphs, B_1 and B_2, are *p-equivalent* if they have isomorphic co-cycle spaces. If B_1 and B_2 are p-equivalent we write

$$B_1 \overset{p}{=} B_2.$$

Remark 2. It is true that two bond graphs are s-equivalent if and only if they are p-equivalent, however we defer the proof of this fact to (**5**), where it will be seen to be a corollary of a matroid result.

For a graph G we gave the definition of the cycle space, $W_{cy}(G)$, in (**1**). Dually, the *co-cycle space*, $W_{co}(G)$, is defined as the subspace of $W(G)$ which consists of the co-cycles and edge-disjoint unions of co-cycles of G. See (**12**) for details of this standard definition, but note that these authors call the co-cycle space the *cutset* *space* of G. Since there is a bijection from a graph G to any dual, G^*, which gives a correspondence between cycles of G and co-cycles of G^*, it is clear that $W_{cy}(G)$ is isomorphic to $W_{co}(G^*)$.

Definition. A bond graph is *co-graphic* if there exists a graph G so that $W_{co}(B)$ is isomorphic to $W_{cy}(G)$. We say that the bond graph and the graph are *co-associated.*

Theorem I

A bond graph is graphic if and only if its dual is co-graphic.

Proof: B is graphic if and only if there is a graph G whose cycle space is isomorphic to $W_{cy}(B)$. But, by definition, $W_{cy}(B) = W_{co}(B^*)$. This means that B^* is co-graphic.

Example 2. We continue the analysis of the bond graph, B, used in Example 1 (see Fig. 1). In that example, we gave bases for the s- and p-spaces of B. Listed below are bases for the cycle, internal and coupling spaces of B:

$$W_{cy}(B): 123, 3456$$

$$W_i(B): o_1o_2$$

$$W_c(B): 3o_1.$$

Next we show bases for the co-cycle and co-coupling spaces of B:

$$W_{co}(B): 134, 12, 45, 46$$

$$W_{cc}(B): 4o_2, 1o_1.$$

The co-internal space has only the zero vector, $\varnothing$. In Fig. 2(a) we give a planar diagram of a graph, G. This G is associated with B, since the cycle space of G, $W_{cy}(G)$, is clearly isomorphic to $W_{cy}(B)$. Also, we see that the co-cycle space of G, $W_{co}(G)$, is isomorphic to $W_{co}(B)$. The augmented graph, G^a, constructed from B, is shown in Fig. 2(b). The internal graph, G_i, and the external graph, $G_e = G$, are coupled together using the coupling information in $W_c(B)$.

The dual bond graph, B^*, is illustrated in Fig. 1(b). We can construct a graph associated with B^*, using the inverse cut and paste method. We show the augmented graph, H^a, in Fig. 2(c) and the associated graph in Fig. 2(d). This graph is the external graph, H_e, of H^a and it is seen to be a dual graph, G^*, of G. Thus we see that G is associated with B and G^* is associated with B^*. Also G^* is co-associated with B and G is co-associated with B^*. The internal graphs are clearly not related as duals and hence neither are the augmented graphs.

Remark 3. The comments made in Example 2 are true in general. We prove in (**5**) that, provided both B and B^* are graphic, then B is associated with G if and only if B^* is associated with G^*, a dual of G. If we consider the augmented graphs, only the external graphs are related as duals. The internal graphs are not related in this manner and hence neither are the augmented graphs.

Remark 4. It is possible for a bond graph to be graphic while its dual is non-graphic. In this case, of course, Theorem I implies that the dual must be co-graphic. Example 3 below is such a bond graph. It is also possible for a bond graph and its

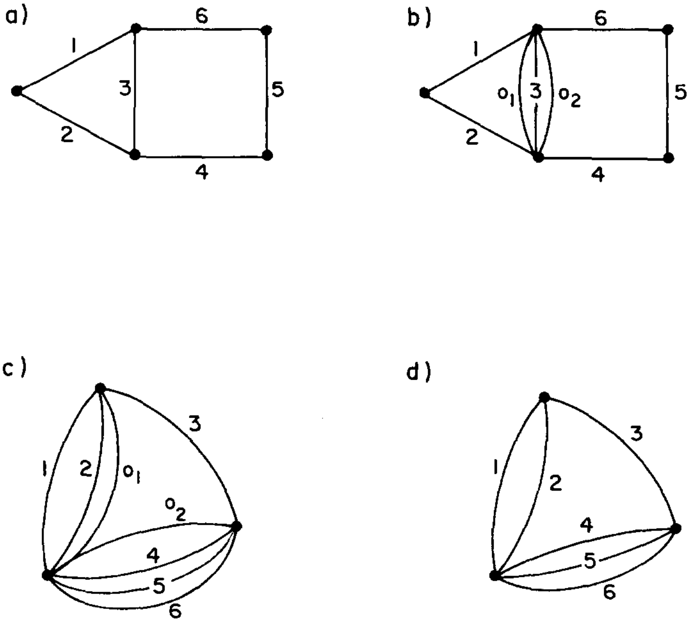

FIG. 2. (a) Graph associated with B. (b) Augmented graph from B. (c) Augmented graph from B^*. (d) Graph associated with B^*.

dual to be non-graphic. An example of a non-graphic bond graph and non-graphic dual is the bond graph in Example 6 of (**1**).

Remark 5. In the application of bond graphs to physical system modelling, the choice of analogy used for the flows and efforts influences whether the bond graph or its dual is used. If we reverse the physical analogy used with a bond graph B, we must use the dual bond graph, B^*, for the new model. Of course, reversing the physical analogy will also necessitate changes to the physical components, but, as noted in (**1**) we do not discuss physical considerations in this theory. Dualizing the analogy requires an inherent dualizing of the combinatorial structure of the physical system. It is this aspect that is relevant in the present discussion. From this point of view it is apparent that a bond graph model produced using the systematic method in a mechanical domain must necessarily use the force-effort analogy. This is because the systematic method of writing bond graph models produces a combinatorial structure which is dual to that produced using a linear graph modelling method. Unless the conventional meaning of the junctions is reversed (which is undesirable since it would only apply in mechanical domain), we must necessarily use the force-effort analogy. Using the abstract theory developed herein, we have shown that the choice of physical analogy is forced upon a modeller for combinatorial reasons and not for physical reasons.

Example 3. The bond graph, B, illustrated in Fig. 3(a) is associated with K_5, the complete graph on 5 vertices, illustrated in Fig. 3(b). The dual bond graph, B^*,

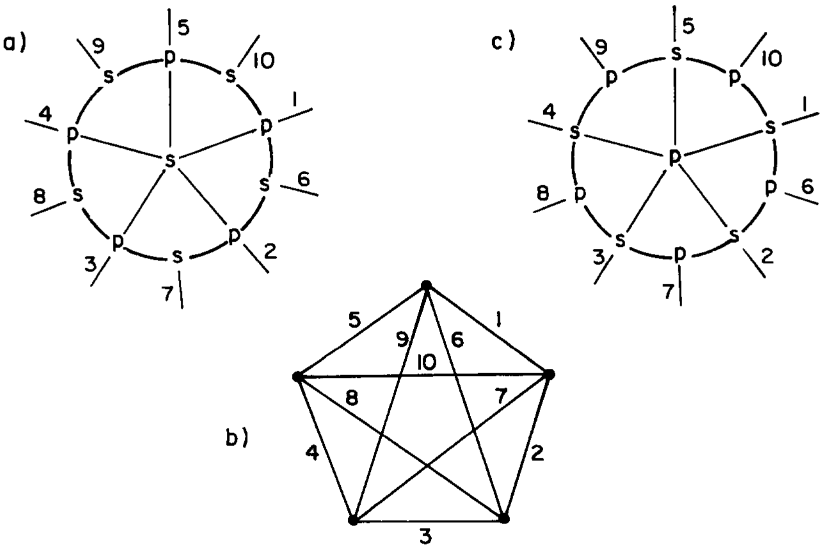

FIG. 3. (a) Graphic bond graph B. (b) Complete graph K_5 associated with B. (c) Non-graphic dual bond graph B.

shown in Fig. 3(c), is non-graphic. B^* is co-graphic, since it is co-associated with K_5.

Remark 6. In most cases a bond graph which contains the combinatorial structure of a physical system will be both graphic and co-graphic. However, it is possible for a physical system to have the structure of a non-planar graph G, like for instance the graph, K_5, of Example 3. In this case the bond graph used for the model cannot be both graphic and co-graphic. It will be either graphic or co-graphic, depending on which physical analogy is used (see Remark 5). This completes the discussion which was begun in Remark 12 of (**1**). A bond graph which is neither graphic nor co-graphic cannot represent a physical system regardless of which physical analogy is used. Such bond graphs are completely different from any previously known examples of "non-physical" bond graphs (see *Introduction*).

IV. s-*Equivalence in the Bond Space*

Definition. Let x and y be vectors in $W(B)$. We shall say that x *is s-equivalent to* y if $x = v+y$ for some $v \in W_s(B)$. If x is s-equivalent to y we write $x \overset{s}{=} y$.

Theorem II

Let B be a simple bond graph and b any bond of B. If x is a vector in $W(B)$ and $x \overset{s}{=} b$ then x is not in $W_s(B)$.

Proof: Suppose that $x \in W_s$ and $x \overset{s}{=} b$. Then $x = v+b$, for some $v \in W_s$, and so $b = x-v$. Now x and v are in W_s which is a subspace and so b is in W_s. But W_s contains no single bond vector, since B is a simple bond graph (in particular B has

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no degenerate s-junction, by definition). Thus we have a contradiction which establishes the result of the theorem.

Corollary. No vector in $W_{cy}(B)$ can be s-equivalent to a single bond vector.

Theorem III

Let B be a bond graph and x, y and z be vectors in $W(B)$. We have the following results:

(i) $x \stackrel{s}{=} \varnothing$ if and only if $x \in W_s$
(ii) $x \stackrel{s}{=} x$
(iii) $x \stackrel{s}{=} y$ if and only if $y \stackrel{s}{=} x$
(iv) $x \stackrel{s}{=} y$ and $y \stackrel{s}{=} z$ imply $x \stackrel{s}{=} z$
(v) $x \stackrel{s}{=} y$ if and only if $x+y \in W_s$
(vi) $x \stackrel{s}{=} y$ if and only if $x+z \stackrel{s}{=} y+z$.

Proof:

(i) $x \stackrel{s}{=} \varnothing \Leftrightarrow x = v+\varnothing,\ v \in W_s \Leftrightarrow x \in W_s$
(ii) $x = \varnothing + x$ and $\varnothing \in W_s$
(iii) $x \stackrel{s}{=} y \Leftrightarrow x = v+y,\ v \in W_s \Leftrightarrow y = -v+x = v+x \Leftrightarrow y \stackrel{s}{=} x$
(iv) $x \stackrel{s}{=} y$ and $y \stackrel{s}{=} z \Leftrightarrow x = v+y$ and $y = v'+z$, with $v, v' \in W_s$
$\Rightarrow x = v+(v'+z) = (v+v')+z$ and $v+v' \in W_s$
(v) $x = v+y \Leftrightarrow x+y = x-y = v \in W_s$
(vi) $x = v+y \Leftrightarrow x+z = (v+y)+z = v+(y+z) \Leftrightarrow x+z \stackrel{s}{=} y+z$.

Statements (ii), (iii) and (iv) of Theorem III establish that the relation of s-equivalence of vectors in W is an equivalence relation. In fact, this is the equivalence relation which is used in linear algebra to define the cosets of vectors in a vector space with respect to a subspace. The *coset* of a vector x is defined by

$$[x] = \{y \in W(B) \mid x \stackrel{s}{=} y\}.$$

We can see that the coset $[x]$ consists of all vectors in W of form $x+v$ with $v \in W_s$, thus the coset is usually denoted by $x+W_s$. We shall follow this practice. Consider the set of all cosets of vectors in W. We can define the *sum* of two cosets by

$$(x+W_s)+(y+W_s) = (x+y)+W_s.$$

Also, we can define the *scalar product* by

$$c(x+W_s) = (cx)+W_s$$

where $c \in GF(2)$. It is easy to check that these cosets are well-defined, meaning that they are independent of the particular representatives chosen from the cosets. Thus we have a vector space structure on the set of all cosets of vectors in W. This vector space (over $GF(2)$) is called the *quotient space* of W with W_s and is denoted by W/W_s. W/W_s is not a subspace of W but it has the property that it is isomorphic to any subspace of W, say W', which is complementary to W_s (i.e. $W = W' \oplus W_s$).

These definitions and results concerning quotient spaces are standard in linear algebra. See, for instance, Hoffman and Kunze (**13**) for further details.

We shall refer to the quotient space, W/W_s, in the discussion of orthogonality in Section V. Also, in (**5**) and (**6**) we use this quotient space more extensively.

Remark 7. We can also define the quotient space W/W_p using cosets with respect to p-equivalence of vectors in $W(B)$. However it is not necessary to consider this dual theory independently, since $W(B)/W_p(B)$ is the same as $W(B^*)/W_s(B^*)$. Thus we only need to develop s-equivalence.

V. Contraction of Bonds

Definition. Let B be a bond graph and suppose that J_1 and J_2 are adjacent junctions of B with the same label. Form a new bond graph, B', by deleting the common (internal) bond joining J_1 and J_2 and identifying the junctions J_1 and J_2 to form a new single junction, J, with the same label as J_1 and J_2. All other bonds on J_1 or J_2 are to be incident on J. We shall say that the common internal bond joining J_1 and J_2 has been *contracted* and the bond graph B' is a *contraction* of bond graph B.

Definition. Let B be a simple bond graph. We now construct a new bond graph, denoted by B_c, from B. If B is proper, define B_c to be B. Otherwise form a sequence of contractions of B by contracting any internal bond which joins junctions with the same label. Eventually we arrive at a proper bond graph which we define to be B_c. We call B_c the *proper contraction* of B.

Theorem IV

A bond graph, B, is s- and p-equivalent to its proper contraction, B_c.

Proof: The conclusion is obvious if $B_c = B$. Suppose on the contrary that B is not proper. We show first that each contraction of a bond common to two junctions with the same label, produces a bond graph which is s-equivalent to B. Then $B \overset{s}{=} B_c$ follows by the transitivity of s-equivalence of bond graphs.

Let v be in $W_{cy}(B)$. Suppose that s_1, the elementary junction vector on s-junction s_1, is included in some elementary decomposition of v. Let s_2 be some other s-junction of B, adjacent to s_1. No internal bond can be in v, since v is in W_{cy}. So s_2, the s-type vector on junction s_2, must also be included in the elementary decomposition of v. Otherwise, the internal bond joining s_1 and s_2 would be in v, since it would not cancel. Now, both s_1 and s_2 are included in the decomposition if and only if s_1+s_2 is included. Thus, if s_1 and s_2 are adjacent s-junctions, an elementary decomposition of v includes s_1 if and only if it includes s_1+s_2.

Let B' be the bond graph formed when the internal bond joining s_1 and s_2, say o, is contracted. The s-type vector on the contracted junction, s, is precisely s_1+s_2 (the only common bond is o, which cancels in this sum). All the other junctions in B' are identical to those of B. Hence we have shown that the vectors in $W_{cy}(B)$ are identical to those of $W_{cy}(B')$. By definition this means that B and B' are s-equivalent.

Suppose, now, that p_1 and p_2 are adjacent p-junctions of B, joined by internal bond o. Let B' be the bond graph formed when o is contracted and denote the contracted junction by p. The p-type vectors of B are identical to those of B', except vectors of form b_1o and b_2o, where b_1 and b_2 are bonds on p_1 and p_2, respectively, and neither of these is o. But, for a vector v in $W_{cy}(B)$, a vector of form b_1o is included in an elementary decomposition of v if and only if a vector of form b_2o is also included. In this case, $b_1o + b_2o = b_1b_2$ is included in the decomposition. Such a vector is a p-type vector for the contracted junction, p, of B'. Hence, we have shown that vectors in $W_{cy}(B)$ are identical to those in $W_{cy}(B')$, when a bond joining two p-junctions is contracted.

Combining the results above and applying the transitivity of s-equivalence of bond graphs, we see that $B \overset{s}{=} B_c$.

Finally we have

$$W_{co}(B) = W_{cy}(B^*) = W_{cy}(B_c^*) = W_{co}(B_c).$$

The middle equality follows because B_c^* is the proper contraction of B^* and we have already established the s-equivalence of the proper contraction. Thus $B \overset{p}{=} B_c$.

Remark 8. Theorem IV is important because it allows us to choose a unique proper bond graph with the same cycle and co-cycle space as a given bond graph. We have established the existence and uniqueness of this contraction, both of which are tacitly assumed in the bond graph literature. Hence, with regard to the algebraic analysis of non-oriented bond graphs we can essentially ignore non-proper bond graphs. In (7) we shall see that a non-proper bond graph can be used to obtain orientation changes which are sometimes useful, for example in the application to physical system modelling.

VI. Orthogonality

Definition. The *inner product* of vectors v and w in $W(B)$ is defined by

$$\langle v \mid w \rangle = \sum_{k=1}^{n} v^k w^k$$

where $v = (v^1, v^2, \ldots, v^n)$ and $w = (w^1, w^2, \ldots, w^n)$ are co-ordinate representations with respect to the standard basis of W.

Definition. We shall say that v and w are *orthogonal* if $\langle v \mid w \rangle = 0$. For a subspace V of $W(B)$ we define the *orthogonal complement*, $V^{\perp}$, to be the set of all w in $W(B)$ so that w is orthogonal to each vector in V. We shall also say that two subspaces V and U are *orthogonal* if each vector in V is orthogonal to each vector in U.

It is easy to show that $V^{\perp}$ is a subspace of W and for any V we have $W = V + V^{\perp}$. However, it is not true in general that $W = V \oplus V^{\perp}$, since $\langle v \mid v \rangle$ may be zero for some v (in particular, this is true for any v with an even number of bonds) and so $V \cap V^{\perp} \neq \emptyset$. Thus the inner product we have defined is a generalized inner product on $W(B)$. The lack of a direct sum orthogonal complement for a subspace of W, in particular for the s-space W_s, is the reason why we must appeal to the concept of quotient space and consider W/W_s instead.

We are now in a position to prove the crucial result contained in Theorem V.

Theorem V

Let B be a simple bond graph. Then $W_{cy}(B)$ is orthogonal to $W_{co}(B)$.

Proof: By virtue of Theorem IV we may assume, without loss of generality, that B is proper. Let $v \in W_{cy}(B)$ and $w \in W_{co}(B) = W_{cy}(B^*)$ be arbitrary vectors. We must show that $\langle v \mid w \rangle = 0$.

Now $v \in W_s$ and $w \in W_p$, thus by Lemma 1 we can decompose v and w as elementary decompositions, v as s- and p-type vectors and w as s^*- and p^*-type vectors. Suppose that one such decomposition is fixed for v and one is fixed for w.

We shall use "p" to denote any one of the p-type vectors on p-junction p and similarly "s^*" will denote any one of the s^*-type vectors on s-junction s.

The inner product is linear and so we can expand $\langle v \mid w \rangle$ using the linear combinations we have chosen. The resulting sum will consist of terms of the following four types:

$$(1) \quad \langle s \mid s^* \rangle$$

$$(2) \quad \langle p \mid p^* \rangle$$

$$(3) \quad \langle s \mid p^* \rangle$$

$$(4) \quad \langle p \mid s^* \rangle.$$

Now we have assumed that B is proper. Thus, by Lemma 2, s and s^* have either 0 or 2 common bonds. In particular there is an even number of common bonds between two vectors of type s and s^*. Hence all terms of type (1) in the expansion above will be zero. A similar application of Lemma 2 to the terms of type (2) shows that all of these terms will be zero.

Consider a pair of junctions of B, one s- and one p-junction. If the two are not adjacent, then, by Lemma 2, terms of type (3) and (4) will be zero, since there are no common bonds between elementary and dual elementary junction vectors on non-adjacent junctions.

Thus, we must consider adjacent s- and p-junctions to determine the non-zero contributions of type (3) and (4) above. Suppose that we have such a pair of junctions, and s and p^* appear in the decompositions chosen for v and w respectively. For this pair of junctions, $\langle s \mid p^* \rangle$ will always be one, by Lemma 2. We shall show that there must be a corresponding term of type (4) in the inner product.

Let o denote the bond joining s and p. This is the only bond common to both junctions. Now v and w do not contain any internal bonds, by definition. Therefore, in the decomposition chosen for v, there must appear a p-type vector, p, on junction p, which contains o. Similarly, in the decomposition chosen for w, there must appear an s^*-type vector, s^*, on s-junction s, which contains o. Now $p \cap s^* = o$, since only the bond o is common to both s and p. These vectors contribute a term of type (4), $\langle p \mid s^* \rangle$, to the inner product. By Lemma 2 we have $\langle p \mid s^* \rangle = 1$.

The argument above shows that, for each non-zero term of type (3) in the inner product, there must be a corresponding non-zero term of type (4). The net contribution is an even number of 1's and this equals zero in $GF(2)$ arithmetic. Suppose now that we have a non-zero contribution of type (4). This occurs only for a pair of adjacent junctions, s and p with common bond o. For a non-zero term of type (4), the junctions p and s must contribute a p-type vector and an s^*-type vector, respectively, to the decompositions of v and w, and furthermore, the bond o must be common to both of these vectors.

Now v is in W_{cy} and so it cannot contain any internal bond. Therefore, included in the decomposition for v, there must be a second vector which contains the bond

o. There are two possibilities for this second vector, either the s-type vector on s or, on p, another p-type vector which contains o. A similar argument, applied to w, shows that the decomposition for w must include either the p^*-type vector on p or, on s, another s^*-type vector which includes o. There are then 4 cases to be considered. Note that the bond o is common to every vector below.

(i) v includes 2 p-type and w includes 2 s^*-type. There will be four terms of type $\langle p \mid s^* \rangle$ and each of these will be one, since each p and s^* vector have only the bond o in common. An even number of ones is zero in $GF(2)$ arithmetic.

(ii) v includes 2 p-type and w includes an s^*-type on s and the p^*-type on p. There will be two terms of type $\langle p \mid s^* \rangle$ and each of these will be one. There are also two terms of type (2), but these are always zero. The net contribution to the inner product is zero.

(iii) v includes a p-type on p and the s-type on s, and w includes two s^*-type on s. Similarly to case (ii), we will have two terms of type $\langle p \mid s^* \rangle$, and both of these will be one. Also there will be two type (1) terms, but these are always zero. The net contribution is zero.

(iv) v includes a p-type on p and the s-type on s, and w includes an s^*-type on s and a p^*-type on p. There will be one term of each of the types (1) to (4) in the expansion of the inner product. The type (1) and type (2) terms will always be zero. We began with a non-zero type (4) term and the type (3) term will be one, since s and p are adjacent. Again we see that the net contribution is zero.

Note that case (iv) is a partial converse for the statement proved above, that a non-zero type (3) term implies the existence of a non-zero type (4) term also. We have now shown that type (3) and type (4) terms are present in the expansion of $\langle v \mid w \rangle$ in such a way that they have no net contribution.

Corollary. The cycle and co-cycle spaces of a graph are orthogonal.

Proof: This result follows by choosing a bond graph $B \in \mathscr{B}(G)$, where G is the graph in question, and application of the theorem to B.

Remark 9. The corollary to Theorem V is a well-known graph-theoretic result [see, for instance, Swamy and Thulasiraman (**12**)]. It is interesting to note, though, that the proof of Theorem V is entirely combinatorial in nature and uses no matrices. An orthogonality result is also known for bond graphs, and this is stated as a property of junction structure matrices [see, for instance, Ort and Martens (**11**)]. This orthogonality property of junction structure matrices is also a corollary of Theorem V. This will be demonstrated when we consider matrix representations of matroids of a bond graph, in (**6**). It is clear, then, that Theorem V is a rather general combinatorial result and neither the statement nor the proof rely on any matrix properties. This non-matrix aspect to the problem is also reflected in the co-ordinate free approach to the vector spaces of the bond graph.

Example 4. We can illustrate the result of Theorem V, by considering the bond graph used in Example 2. It is clear that each basis vector of $W_{cy}(B)$ is orthogonal to each basis vector of $W_{co}(B)$. For instance we show 3 sample calculations below:

$$\langle 123 \mid 134 \rangle = 1+1 = 0$$
$$\langle 123 \mid 45 \rangle = 0$$
$$\langle 3456 \mid 46 \rangle = 1+1 = 0.$$

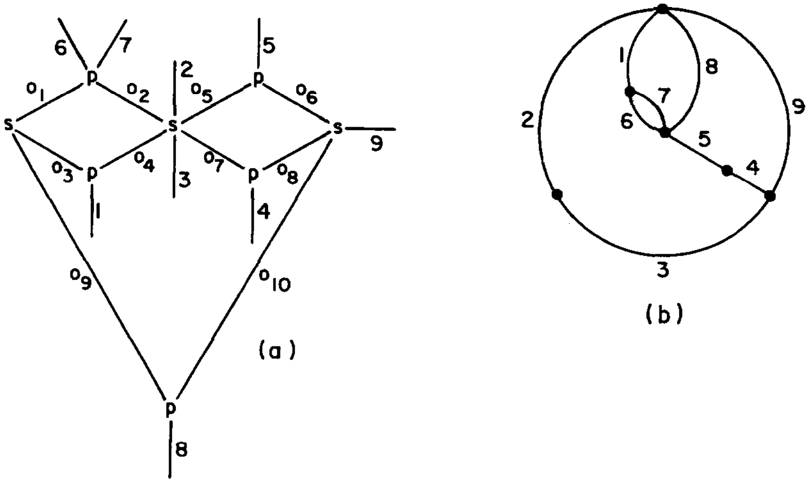

FIG. 4. (a) Bond graph B_1. (b) Associated graph planar representation G_1.

VII. Example

For a bond graph, B, we can easily find a basis for the co-cycle space, by writing a p^*-type vector for each p-junction and eliminating internal bonds systematically, by using other elementary junction vectors. We will have constructed a basis for $W_{co}(B)$, after removing any redundant vectors in the list and adding the appropriate independent s^*-type vectors which involve only external bonds. A similar procedure using s- and p-type vectors, will produce a basis for the cycle space of B. This technique is quite simple to apply, even to a rather complex bond graph. We illustrate this procedure in the following example, by constructing the co-cycle space of a bond graph.

Example 5. Consider the bond graph, B_1, shown in Fig. 4(a). We shall find the co-cycle space of B_1. There are 4 p-junctions which contribute independent vectors:

$$67o_1o_2+o_1o_3+o_3 1o_4+o_4o_2 = 167$$

$$5o_5o_6+o_5 2+o_6 9 = 259$$

$$8o_9o_{10}+o_9o_3+o_{10}9+o_3 1o_4+o_4 3 = 1389$$

$$4o_7o_8+o_7 2+o_8 9 = 249.$$

Combining these vectors with the s^*-type vector 23, produces the basis

$$\{167, 259, 1389, 249, 23\}$$

for $W_{co}(B_1)$.

This co-cycle space is isomorphic to the co-cycle space of the graph, G, for which Fig. 4(b) shows a planar representation, G_1. G is associated with the bond graph, B_1, which was constructed from the planar representation G_1, using the cut and paste method.

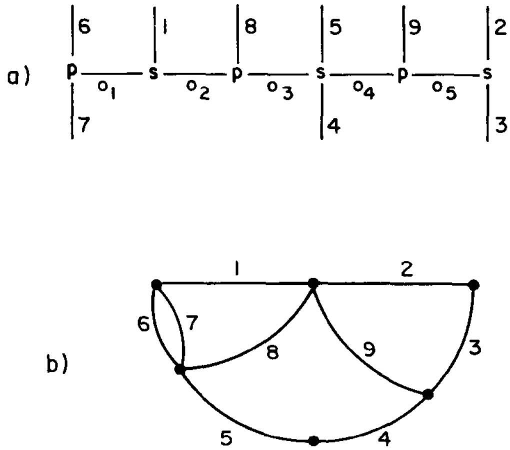

FIG. 5. (a) Bond graph B_2. (b) Associated graph planar representation G_2.

Now consider the bond graph, B_2, shown in Fig. 5(a). This is the bond graph obtained from a different planar representation, G_2, of the graph G, using the cut and paste method. The representation G_2 is shown in Fig. 5(b). If we follow the same technique, we obtain 3 independent vectors in $W_{co}(B_2)$ from the 3 p-junctions of B_2:

$$67o_1+o_11 = 167$$

$$9o_4o_5+o_44+o_52 = 249.$$

There are 2 independent s^*-type vectors, 45 and 23. Combining these with the 3 above, we obtain the basis

$$\{167, 1348, 249, 23, 45\}$$

for $W_{co}(B_2)$. This basis generates precisely the same space as the one given above for B_1. Thus the two bond graphs are p-equivalent.

It is easy to show that B_1 and B_2 are also s-equivalent bond graphs.

In Fig. 6 we show a planar representation of a dual graph, G^*, for G. The dual bond graphs B_1^* and B_2^* are associated with G^*. Also B_1 and B_2 are co-associated with this dual graph G^*.

VIII. Conclusions

The dual bond graph, B^*, of a bond graph, B, has been defined. This led to a discussion of various dual concepts for B. For instance, the p-space, $W_p(B)$, is the dual of the s-space, $W_s(B)$. W_p has a decomposition into the co-cycle, co-internal and co-coupling spaces of B. In particular the co-cycle space, $W_{co}(B)$, the dual of the cycle space, $W_{cy}(B)$, has many interesting properties.

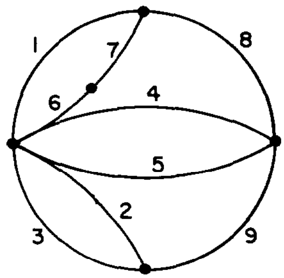

FIG. 6. Planar representation of dual graph, G^*.

A graph, G, and a bond graph, B, are co-associated if $W_{co}(B)$ is isomorphic to $W_{cy}(G)$, the cycle space of G. This is the dual of the concept of association of a graph and a bond graph. Bond graphs which have a co-associated graph are called co-graphic. Examples have been given illustrating that for a graph, G, and an associated bond graph B, a dual graph, G^*, is associated with the dual bond graph, B^*. A general result on dual graphs has been deferred to (5).

If B is a non-proper bond graph, the proper contraction of B, B_c, is a simplified version of B which has the same cycle and co-cycle space. The contraction of bonds joining similarly labelled junctions is commonly used in the literature of bond graph modelling, and we have established the justification for the topological significance of the procedure.

The co-cycle space and cycle space of a bond graph have been shown to be orthogonal. This implies the well-known orthogonality relationship for a graph and also for junction structure matrices.

A simple procedure has been explained for constructing a basis for either the cycle or the co-cycle space of a bond graph. This technique uses the bond graph directly.

We have demonstrated that the choice of physical analogy used in a bond graph model of a physical system is forced upon a modeller for combinatorial reasons (see Remark 5). Thus, there is no basis to any statement that one analogy is in some sense preferable to the other.

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Appendix. Duality for Graphs

Let G_p be a planar representation of a graph G. We now describe how to construct a dual planar representation, G_p^*, called the *geometric dual* of G_p. We choose one point in each face of G_p, including the unbounded face, and these points are the vertices of the geometric dual. For each edge of G_p which is common to the closures of two faces, we draw an edge of G_p^* joining the corresponding vertices and through the common edge. This is a standard procedure and is discussed in detail in Welsh (**14**).

A different concept of duality for graphs is also used. This does not use planar representations and so it is defined algebraically. A graph G^* is an *abstract dual* of a graph G if there is a bijection between the edge sets of G and G^* so that a cycle of G corresponds to a co-cycle of G^*.

For planar graphs it can be shown that every geometric dual is also an abstract dual but the converse is false. Thus every planar graph has an abstract dual. Clearly, in this case, G must be an abstract dual of G^*. See Welsh (**14**) for proofs of these statements and for examples.